

CALABI-YAU ALGEBRAS AND SUPERPOTENTIALS

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ABSTRACT. We prove that complete d -Calabi-Yau algebras in the sense of Ginzburg are derived from superpotentials.

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1. INTRODUCTION

In this introduction we assume that k is an algebraically closed field of characteristic zero. In the foundational paper [13] Ginzburg defines a k -algebra A satisfying suitable finiteness conditions to be d -Calabi-Yau if there is a quasi-isomorphism of A -bimodules

$$\eta : \mathrm{RHom}_{A^e}(A, A \otimes A) \xrightarrow{\cong} \Sigma^{-d} A$$

This property implies for example that the category of finite dimensional A -modules is d -Calabi-Yau in the usual sense. Sometimes one imposes the additional condition that η is self dual but this appears to be automatic. See Appendix C.

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In loc. cit. Ginzburg also introduces a particular class of 3-Calabi-Yau algebras which has found many applications in the theory of cluster algebras [10, 11] and cluster categories [1, 17, 20]. Let $Q = (Q_0, Q_1)$ be a finite quiver with vertices Q_0 and arrows Q_1 . By definition a *superpotential* is an element w of $kQ/[kQ, kQ]$. The *Ginzburg algebra* $\Pi(Q, w)$ is the DG-algebra $(k\bar{Q}, d)$ where \bar{Q} is the graded quiver with vertices Q_0 and arrows

- the original arrows a in Q_1 (degree 0);
- opposite arrows a^* for $a \in Q_1$ (degree -1);
- loops z_i at vertices $i \in Q_0$ (degree -2).

The differential is¹

$$\begin{aligned} da &= 0 & (a \in Q_1) \\ da^* &= -\frac{\circ\partial w}{\partial a} & (a \in Q_1) \\ dz &= \sum_{a \in Q_1} [a, a^*] \end{aligned}$$

for $z = \sum_i z_i$. Here $\circ\partial/\partial a$ is the so-called *circular derivative*

$$\frac{\circ\partial w}{\partial a} = \sum_{w=uv} vu$$

The homology in degree zero of a Ginzburg algebra is a so-called *Jacobi algebra*. Ginzburg shows that if the homology of $\Pi(Q, w)$ is concentrated in degree zero then the associated Jacobi-algebra $H^0(\Pi(Q, w))$ is 3-Calabi-Yau. In [17] a DG-version of this result is proved.

As observed in [13, §3.6] the definition of $\Pi(Q, w)$ is not tied to $d = 3$ (see also Lazaroiu's work in [23]). Let Q be an arbitrary finite graded quiver and let \bar{Q} be the corresponding *double quiver* obtained from Q by adjoining opposite arrows a^* of degree $-d + 2 - |a|$ for $a \in Q_1$ (there is a slight subtlety with loops which we gloss over, see §10.3). It is now well-known that $N = k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ is a Lie algebra when equipped with the so-called *necklace bracket* $\{-, -\}$ [3, 14, 22]. Let $w \in N$ be such that $|w| = -d + 3$ and $\{w, w\} = 0$ and let \tilde{Q} be obtained from \bar{Q} by adjoining loops $(z_i)_i$ of degree $-d + 1$ as above. Then the *deformed DG-preprojective algebra*² $\Pi(Q, d, w)$ is the DG-algebra $(k\tilde{Q}, d)$ with differential

$$\begin{aligned} da &= (-1)^{(|a|+1)|a^*|} \frac{\circ\partial w}{\partial a^*} & (a \in Q_1) \\ da^* &= -(-1)^{|a|} \frac{\circ\partial w}{\partial a} & (a \in Q_1) \\ dz &= \sum_{a \in Q_1} [a, a^*] \end{aligned}$$

Again Ginzburg proves that if the homology of $\Pi(Q, d, w)$ is concentrated in degree zero then $H^*(\Pi(Q, d, w))$ is d -Calabi-Yau.

Results about Calabi-Yau algebras are often most conveniently proved under the hypothesis that the algebras are derived from superpotentials (see e.g. [8, 9]). It is therefore a natural question how restrictive this hypothesis is. Before dealing with

¹The unusual sign in the definition of da^* is an artifact of our setup.

²This is our own terminology.

this we note that it is generally understood that the above definition of Calabi-Yau should somehow be strengthened to include higher homotopy information in the definition of η .

A suitable strengthening of the Calabi-Yau property was suggested to the author by Bernhard Keller. It is related to a dual property used by Kontsevich and Soibelman in [21].

We first observe that η can be interpreted as a class in the Hochschild homology group $\mathrm{HH}_d(A)$. Then we make the following definition

Definition. A d -Calabi-Yau algebra A is *exact Calabi-Yau*³ if η can be taken in the image of the Connes map $B : \mathrm{HC}_{d-1}(A) \rightarrow \mathrm{HH}_d(A)$.

Even with this strengthening of the Calabi-Yau property, it is probably only sensible to attempt a classification in sufficiently local cases. In this paper we will discuss the *complete* case. That is, roughly speaking, we discuss topological algebras which are quotients of quivers completed at path length. For technical background see §4 and also [20, Appendix]. Note that the complete case encompasses the graded case which has been treated in [2]. Indeed the category of graded algebras is equivalent to the category of complete algebras equipped with a k^* -action.

The main results in this paper are the following.

Theorem A. (see Corollary 9.3) A “complete” d -Calabi-Yau algebra is exact d -Calabi-Yau.

This result depends on a vanishing property for periodic cyclic homology which follows from Goodwillie’s classical result for nilpotent extensions.

Theorem B. (see Theorem 10.2.2 and §10.3) A “complete” exact d -Calabi-Yau DG-algebra A concentrated in degrees ≤ 0 such that $A/\mathrm{rad} A$ is commutative, is quasi-isomorphic to a (completed) deformed DG-preprojective algebra $\Pi(Q, d, w)$ with the degrees of the arrows in Q lying in the interval $[(-d+2)/2, 0]$ and w being a linear combination of paths of length at least 3.

By combining Theorems A and B it follows in particular that a complete 3-Calabi-Yau algebra is a Jacobi algebra. This result has been announced several years ago by Rouquier and Chuang but so far the proof has not been published. A proof in the graded case for algebras generated in degree one has been given in [2]. A proof has also been given by Ed Segal under a suitable strengthening of the Calabi-Yau condition (see [30, Theorem 3.3]).

Some of the ideas of this manuscript have been used by Davison in [7] where he shows that group algebras of compact hyperbolic manifolds of dimension greater than one are *not* derived from superpotentials.

We now give an outline of the content of this paper. Somewhat arbitrarily it is divided into a main body and appendices. Whereas in the body of the paper we often impose boundedness conditions on our DG-algebras, and furthermore k is often characteristic zero, we have avoided making such restrictions in the appendices.

In §4-6 we discuss pseudo-compact algebras, modules, bimodules, . . . and their homological algebra. Our approach is somewhat different from [20, Appendix] as it relies heavily on duality. We also need the bar cobar formalism which to the best

³This is our own terminology.

of the author's knowledge has not been systematically developed in the pseudo compact setting (although this turns out to be easy). In order to make the text not too heavy we have deferred most details to Appendix A.

In §7 we briefly discuss cyclic homology and its extension to the pseudo-compact case. We also remind the reader of the X -complex formalism due to Cuntz and Quillen.

In §8 we discuss the different notions of Calabi-Yau algebras and in §9 we prove Theorem A.

In §10 we introduce deformed DG-preprojective algebras and in §11 we prove Theorem B.

In §12 we prove that the Koszul dual of a pseudo-compact exact Calabi-Yau algebra has a cyclic A_∞ -structure.

In Appendix B we prove the technical result that the pseudo-compact Hochschild complex really computes $A \overset{L}{\otimes}_{A^e} A$.

In Appendix C we prove that the morphism η appearing in Ginzburg's definition of a Calabi-Yau algebra is automatically self dual.

Finally in Appendix D we prove some results on the behaviour of Hochschild/cyclic homology under Koszul duality.

2. ACKNOWLEDGEMENT

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This paper was furthermore strongly influenced by ideas of Ginzburg [13], Kontsevich and Soibelman [21] and Lazaroiu [23].

The author thanks Maxim Kontsevich for pointing out to him that "exact" Calabi-Yau is a better terminology than "strongly" Calabi-Yau which was used in the first version of this article.

3. NOTATION AND CONVENTIONS

Throughout k will be a ground field. Unadorned tensor products are over k . We follow a version of the Sweedler convention in the sense that an element a of a tensor product is written as $a' \otimes a''$ (without summation sign).

Unless otherwise specified algebras have units and maps are unit preserving. Modules are unital as well. A similar convention holds for coalgebras and comodules.

If we write $a \in A$ for A graded then we implicitly assume that a is homogeneous. The degree of a is written as $|a|$. The shift functor on complexes is written as Σ . The elements of ΣA are written as sa for $a \in A$ with $|sa| = |a| - 1$. $\text{Hom}_A(-, -)$ denotes graded Hom's. If we only want degree zero maps we write $\text{Hom}_{\text{Gr}(A)}(-, -)$.

To reduce clutter all operations are implicitly completed when working with pseudo-compact objects. Indeed the completions are implicit in the monoidal structure on the underlying category of pseudo-compact vector spaces.

Throughout Δ means comultiplication, μ means multiplication, ϵ is either the counit or coaugmentation and η is either the unit or augmentation.

The 1-form associated to a function f is denoted by Df . This is to avoid confusion with the differential on DG-objects which will be denoted by d . We regard

$D(-)$ as an operation of homological degree zero. In other words it commutes with d .

4. PSEUDO-COMPACT OBJECTS

When dealing with completed path algebras the natural context to work in is that of pseudo-compact vector spaces/rings/modules etc. . . . See [12, 20, 31]. We briefly recall the salient features of this setting.

By definition a *pseudo-compact k -vector space* is a linear topological vector space which is complete and whose topology is generated by subspaces of finite codimension. We will denote the corresponding category by $\text{PC}(k)$. The topology on finite dimensional pseudo-compact vector spaces is necessarily discrete and conversely a finite dimensional vector space with the discrete topology is pseudo-compact. In particular k itself is naturally pseudo-compact.

We have inverse dualities

$$(4.1) \quad \begin{aligned} \mathbb{D} : \text{Mod}(k) &\rightarrow \text{PC}(k)^\circ : V \mapsto \text{Hom}(V, k) \\ \mathbb{D} : \text{PC}(k) &\rightarrow \text{Mod}(k)^\circ : W \mapsto \text{Hom}_{\text{cont}}(W, k) \end{aligned}$$

where we recall that for $V \in \text{Mod}(k)$ the topology on $\mathbb{D}V$ is generated by the kernels of $\mathbb{D}V \rightarrow \mathbb{D}V'$ where V' runs through the finite dimensional subspaces of V .

It follows that $\text{PC}(k)$ is a coGrothendieck category. In particular $\text{PC}(k)$ has exact filtered inverse limits (axiom AB5*). Furthermore since the dual of $\text{PC}(k)$ is locally noetherian, the product of projectives in $\text{PC}(k)$ is projective.

One checks that the functor forgetting the topology

$$\text{PC}(k) \rightarrow \text{Mod}(k)$$

is exact, faithful and commutes with inverse limits. However it does not commute with infinite direct sums.

We will systematically use \mathbb{D} to transfer notions from $\text{Mod}(k)$ to $\text{PC}(k)$. Thus if $V, W \in \text{PC}(k)$ then we put

$$V \otimes W = \mathbb{D}(\mathbb{D}W \otimes \mathbb{D}V)$$

or concretely

$$V \otimes W = \text{proj} \lim_{V', W'} V/V' \otimes W/W'$$

where V', W' run through the open subspaces of V, W respectively. By construction \mathbb{D} is compatible with the monoidal structure.

Below we also need *graded pseudo-compact vector spaces*. These are simply \mathbb{Z} -graded objects in the category $\text{PC}(k)$, i.e. sequences of pseudo-compact objects $(V^i)_{i \in \mathbb{Z}}$ (sometimes written as a formal direct product $\prod_{i \in \mathbb{Z}} V^i$). We denote the corresponding category by $\text{PCGr}(k)$. Putting $\mathbb{D}((V^i)_{i \in \mathbb{Z}}) = (\mathbb{D}V^{-i})_{i \in \mathbb{Z}}$ defines a duality between $\text{PCGr}(k)$ and $\text{Gr}(k)$, the latter being the category of graded k -vector spaces.

There is a natural functor “forgetting the grading” which commutes with \mathbb{D}

$$(-)^u : \text{PCGr}(k) \rightarrow \text{PC}(k) : (V_i)_i \mapsto \prod_i V_i$$

The monoidal structure on $\text{PCGr}(k)$ is given by

$$\begin{aligned} (V_i)_i \otimes (W_j)_j &= \mathbb{D}(\mathbb{D}((W_j)_j) \otimes \mathbb{D}((V_i)_i)) \\ &= \left(\prod_{i+j=k} V_i \otimes W_j \right)_k \end{aligned}$$

Using the monoidal structure on $\text{PC}(k)$ it is possible to define *pseudo-compact k -algebras, modules, bimodules*, etc... which are simply the corresponding objects in $\text{PC}(k)$.

Lemma 4.1. (1) *The topology on a pseudo-compact k -algebra A is generated by twosided ideals of finite codimension.*
 (2) *Let M be a pseudo-compact left module over A . Then the topology on M is generated by submodules of finite codimension.*
 (3) *Let M be a pseudo-compact bimodule over A . Then the topology on M is generated by subbimodules of finite codimension.*

Proof. One may prove this directly using the continuity properties of the multiplication. Alternatively one may use the following observation: if A is a pseudo-compact A -algebra then $\mathbb{D}A$ is a coalgebra. (1) then follows from the fact that a coalgebra is locally finite (every coalgebra is a filtered direct limit of finite dimensional coalgebras). (2)(3) are proved in the same way. \square

Remark 4.2. A pseudo-compact k -algebra A is traditionally defined as a linear topological k -algebra which is complete and whose topology is generated by left ideals of finite codimension [12, 20, 31]. This equivalent to the above definition. Indeed such an A is an object in $\text{PC}(k)$ and as the multiplication on A is continuous it represents an algebra object in $\text{PC}(k)$. Similar observations hold for modules and bimodules.

Let A be a pseudo-compact ring. The common annihilator of the simple pseudo-compact A -modules is called the radical of A and is denoted by $\text{rad } A$. We recall the following

Proposition 4.3. [12, Prop. IV.13] *The radical of A coincides with the ordinary Jacobson radical of A .*

To eliminate another source of confusion we include the following lemma.

Lemma 4.4. *If A is a graded pseudo-compact k -algebra then A^u is an ordinary pseudo-compact algebra and its topology is generated by ideals of the form L^u where L is a graded ideal in A of finite total codimension. We also have*

$$(\text{rad } A)^u = \text{rad}(A^u)$$

Proof. This follows by applying \mathbb{D} to the standard dual statement for coalgebras. \square

If A is a pseudo-compact ring then the category of left pseudo-compact A -modules is denoted by $\text{PC}(A)$. The category of pseudo-compact A -bimodules is usually denoted by $\text{PC}(A^e)$ with $A^e = A \otimes_k A^\circ$. We use similar concepts and notations in the graded context.

A *DG-algebra A over $\text{PC}(k)$* is said to be *pseudo-compact* if it is pseudo-compact as a graded algebra and the differential is continuous. DG-(bi)modules are defined similarly.

Remark 4.5. The topology on a pseudo-compact DG-algebra A has a basis given by twosided DG-ideals of finite codimension. To see this consider the dual statement for coalgebras. This says that any DG-coalgebra (C, d) should be the union of finite dimensional DG-coalgebras.

Indeed if C' is a finite dimensional graded sub-coalgebra in C then $C' + dC'$ is a finite dimensional sub-DG-coalgebra.

Similar comments can be made about left modules and bimodules.

5. DUALITY FOR MODULES/BIMODULES

Below we will fix a finite dimensional separable k -algebra l which will be used throughout as a ground ring. As l is non-commutative this creates some technical problems with duals. As pointed out in [4] there are 4 sensible ways to define the dual of an l -bimodule. However as shown in loc. cit. these can all be identified by fixing a *trace* on l , that is, a k -linear map $\text{Tr} : l \rightarrow k$ such that the bilinear form $(a, b) \mapsto \text{Tr}(ab)$ is symmetric and non-degenerate.

For $V \in \text{Mod}(k)$ put $V^* = \text{Hom}_k(V, k)$. By functoriality it is clear that $(-)^*$ sends left l -modules to right l -modules and vice versa. The challenge is to make the $(-)^*$ -operation compatible with the monoidal structure given by the tensor product over l . It is explained in [4] how to do this.

For the rest of this paper we fix a trace $\text{Tr} : l \rightarrow k$ and we let $\sigma = \sigma' \otimes \sigma'' \in l \otimes_k l$ be the corresponding Casimir element. Note that Tr is unique up to multiplication with a central unit in l . The following properties will be used frequently: for $a \in l$ we have

$$\begin{aligned}\sigma' \otimes \sigma'' &= \sigma'' \otimes \sigma' \\ a\sigma' \otimes \sigma'' &= \sigma' \otimes \sigma''a\end{aligned}$$

We define morphisms of l -bimodules

$$\pi : l^* \rightarrow l : \pi(\phi) = \sigma' \phi(\sigma'')$$

$$c_{W,U} : W^* \otimes_l U^* \rightarrow (U \otimes_l W)^* : \phi \otimes \theta \mapsto (u \otimes w \mapsto \theta(u\sigma')\phi(\sigma''w))$$

for U a right l -module and W a left l -module. The latter morphism is natural in U and W . We claim that these maps are compatible with tensor product as expressed in the following lemma.

Lemma 5.1. *π is a bimodule isomorphism and furthermore the following diagrams are commutative*

$$\begin{array}{ccc} W^* \otimes_l l^* & \xrightarrow{c_{W,l}} & (l \otimes_l W)^* \\ \text{id}_{W^*} \otimes \pi \downarrow & & \parallel \\ W^* \otimes_l l & \xlongequal{\quad} & W^* \end{array} \quad \begin{array}{ccc} l^* \otimes_l U^* & \xrightarrow{c_{l,U}} & (U \otimes_l l)^* \\ \pi \otimes \text{id}_{U^*} \downarrow & & \parallel \\ l \otimes_l U^* & \xlongequal{\quad} & U^* \end{array}$$

In addition if V is an l -bimodule then the following diagram is commutative

$$\begin{array}{ccc} W^* \otimes_l V^* \otimes_l U^* & \xrightarrow{c_{W,V} \otimes \text{id}_{U^*}} & (V \otimes_l W)^* \otimes_l U^* \\ \text{id}_{W^*} \otimes c_{V,U} \downarrow & & \downarrow c_{V \otimes_l W, U} \\ W^* \otimes_l (U \otimes_l V)^* & \xrightarrow{c_{W,U \otimes_l V}} & (U \otimes_l V \otimes_l W)^* \end{array}$$

The maps π and $c_{??}$ are compatible with the canonical linear maps $V \rightarrow V^{**}$ in the following sense.

Lemma 5.2. *The following diagrams are commutative.*

$$\begin{array}{ccc}
 l^* & \xrightarrow{\pi^*} & l^{**} \\
 & \searrow \pi & \nearrow \\
 & l &
 \end{array}
 \quad
 \begin{array}{ccc}
 U^{**} \otimes_l W^{**} & \xrightarrow{c_{U^*, W^*}} & (W^* \otimes_l U^*)^* \\
 \uparrow & & \uparrow c_{W, U}^* \\
 U \otimes_l W & \longrightarrow & (U \otimes_l W)^{**}
 \end{array}$$

We will use these results mostly in the following sense.

Proposition 5.3. *The contravariant functors denoted by \mathbb{D} define inverse dualities as monoidal categories between $\text{PC}(l^e)$ and $\text{Mod}(l^e)$. These dualities are compatible with the appropriate actions on categories of left and right l -modules.*

For an l -bimodule U we define $U_l = U/[l, U]$ and we let U^l be the l -centralizer in U . It is easy to see that the map

$$(5.1) \quad (-)^\S : U_l \rightarrow U^l : m \mapsto \sigma' m \sigma''$$

is an isomorphism of k -vector spaces. We denote its inverse by $(-)^\dagger$,

The operations $(-)_l$, $(-)^l$ are compatible with dualizing in the sense that we have canonical identifications

$$(U_l)^* = (U^*)^l \quad (U^l)^* = (U^*)_l$$

Let U, V be l -bimodules. Then

$$U \otimes_{l^e} V \cong (U \otimes_l V)_l$$

We have a canonical “flip” isomorphism

$$(5.2) \quad \mathfrak{B} : (U \otimes_l V)_l \rightarrow (V \otimes_l U)_l : m \otimes n \mapsto n \otimes m$$

We would like to understand how this plays with duality. We have a map

$$(U^* \otimes_l V^*)_l \xrightarrow{c_{U, V}} ((V \otimes_l U)^*)_l \cong ((V \otimes_l U)^l)^*$$

Now it is not so clear from this how the flip acts on $((V \otimes_l U)^l)^*$. Another way of saying this is that if we look at the pairing derived from $c_{U, V}$

$$(5.3) \quad (U^* \otimes_l V^*)_l \times (V \otimes_l U)^l : (\phi \otimes \theta, v \otimes u) \mapsto \phi(u\sigma')\theta(\sigma''v)$$

then we see that exchanging ϕ and θ is not compensated for by simply exchanging u and v .

However if we compose further

$$(U^* \otimes_l V^*)_l \rightarrow ((V \otimes_l U)^l)^* \xrightarrow{(-)^\S} ((V \otimes_l U)_l)^*$$

then one verifies that the following diagram is commutative

$$\begin{array}{ccc}
 (U^* \otimes_l V^*)_l & \longrightarrow & ((V \otimes_l U)_l)^* \\
 \mathfrak{B} \downarrow & & \downarrow \mathfrak{B}^* \\
 (V^* \otimes_l U^*)_l & \longrightarrow & ((U \otimes_l V)_l)^*
 \end{array}$$

Another way of saying this is that we have a well defined pairing

$$(5.4) \quad (U^* \otimes_l V^*)_l \times (V \otimes_l U)_l : (\phi \otimes \theta, v \otimes u) \mapsto \phi(\sigma'_2 u \sigma'_1) \theta(\sigma''_1 v \sigma''_2)$$

which is now symmetric under simultaneously exchanging (ϕ, θ) and (u, v) .

6. HOMOLOGICAL ALGEBRA IN THE PSEUDO-COMPACT CASE

An *augmented l -algebra* is an algebra A equipped with k -algebra homomorphisms $l \xrightarrow{\eta} A \xrightarrow{\epsilon} l$ such that $\epsilon\eta$ is the identity. We have a corresponding decomposition $A = l \oplus \bar{A}$ where $\bar{A} = \ker \epsilon = \operatorname{coker} \eta$. Note that an augmented l -algebra is entirely determined by the algebra structure on \bar{A} (which is l -linear but likely without unit).

Obviously there are similar concepts in the graded and DG-setting as well as in the coalgebra setting.

Below we use the notation $\operatorname{Alg}(l)$ for the category of augmented l -DG-algebras and $\operatorname{Cog}(l)$ for the category of augmented l -DG-coalgebras. An object C in $\operatorname{Cog}(l)$ is said to be *cocomplete* if l identifies with the coradical of C , i.e. the sum of all simple coalgebras⁴ in C . We denote the full subcategory of $\operatorname{Cog}(l)$ consisting of cocomplete coalgebras by $\operatorname{Cogc}(l)$. We use the notations $\operatorname{PCAlg}(l)$, $\operatorname{PCAlg}(l)$, $\operatorname{PCCog}(l)$ for the corresponding pseudo-compact notions, the definition of which is dual to those of the non-topological notions (we can deduce their definition by applying the functor \mathbb{D}). For example an l -DG-algebra $A \in \operatorname{PCAlg}(l)$ is in $\operatorname{PCAlg}(l)$ iff $\bar{A} = \operatorname{rad} A$.

For $C \in \operatorname{Cog}(l)$ let $\operatorname{DGComod}(C)$ be the category of l -DG-comodules over C . For $A \in \operatorname{Alg}(l)$ let $\operatorname{DGMod}(A)$ be the category of DG-modules over A . Write $\operatorname{PCDGComod}(C)$ and $\operatorname{PCDGMod}(A)$ for the corresponding pseudo-compact notions.

All these categories are equipped with model structures and connected with each other through Quillen equivalences [16, 19, 24, 27]. See Appendix A for a survey and further details. Let $A \in \operatorname{PCAlg}(l)$. The model structure on $\operatorname{PCDGMod}(A)$ is dual to [27, §8.2]. One has

- (1) The *weak equivalences* are the morphisms with an acyclic cone.
- (2) The *cofibrations* are the injective morphisms with cokernel which is projective when forgetting the differential.
- (3) The *fibrations* are the surjective morphisms.

An object is *acyclic* if it is in the smallest subcategory of the homotopy category of A which contains the total complexes of short exact sequences and is closed under arbitrary products. There is also a characterization of weak equivalences in terms of the bar construction. See Appendix A.4.

A weak equivalence between objects in $\operatorname{PCAlg}(l)$ is strictly stronger than a quasi-isomorphism. A similar statement holds for weak equivalences in $\operatorname{PCDGMod}(A)$. However under suitable boundedness assumptions (algebras concentrated in degrees ≤ 0 and modules concentrated in degrees $\leq N$) weak equivalence is the same as quasi-isomorphism (see the dual statements to Proposition A.1.2 and Lemma A.2.1).

7. CYCLIC HOMOLOGY

7.1. Mixed complexes and Hochschild/cyclic homology. We recall that a *mixed complex* is a graded vector space U equipped with maps $b, B : U \rightarrow U$ such that $|b| = 1$, $|B| = -1$, $b^2 = 0$, $B^2 = 0$, $bB + Bb = 0$ [25].

⁴The coradical is automatically graded and equates the graded coradical.

The *Hochschild homology* $\mathrm{HH}_*(U)$, *negative cyclic homology* $\mathrm{HC}_*^-(U)$, *periodic cyclic homology* $\mathrm{HC}_*^{\mathrm{per}}(U)$ and *cyclic homology* $\mathrm{HC}_*(U)$ of U are defined as the homologies of the following complexes

$$\begin{aligned} C(U) &= (U, b) \\ CC^-(U) &= (U[[u]], b + uB) \\ CC^{\mathrm{per}}(U) &= (U((u)), b + uB) \\ CC(U) &= (U((u))/U[[u]], b + uB)[-2] \end{aligned}$$

where u is a formal variable of degree $+2$. We see that cyclic homology is computed as the homology of a *sum total complex* of a suitable double complex whereas negative cyclic homology is obtained from a *product total complex*. Periodic cyclic homology is derived from a total complex which is a mixture between a sum and a product total complex.

By definition a morphism between mixed complexes $(U, b, B) \rightarrow (V, b, B)$ is a *quasi-isomorphism* if $(U, b) \rightarrow (V, b)$ is a quasi-isomorphism. The homotopy category of mixed complexes is obtained by inverting quasi-isomorphisms. Since both products and sums are exact functors it is clear that Hochschild homology and the different variants of cyclic homology remain invariant under quasi-isomorphisms of mixed complexes.

Let A be a unital l -DG-algebra. Then the *Hochschild mixed complex* of A is $(C(A), b, B)$ where $(C(A), b)$ is the (*sum*) total complex of the standard Hochschild double complex

$$(7.1) \quad \cdots \xrightarrow{\partial} (A \otimes_l A \otimes_l A)_l \xrightarrow{\partial} (A \otimes_l A)_l \xrightarrow{\partial} A_l \rightarrow 0$$

As usual B denotes the Connes differential. The *normalized Hochschild mixed complex* $(\bar{C}(A), b, B)$ is [25, §2.1.9] the sum total complex of

$$\cdots \xrightarrow{\partial} (A \otimes_l \bar{A} \otimes_l \bar{A})_l \xrightarrow{\partial} (A \otimes_l \bar{A})_l \xrightarrow{\partial} \bar{A}_l \rightarrow 0$$

for $\bar{A} = A/l$. The ordinary and normalized Hochschild mixed complex are quasi-isomorphic (see [25, Prop. 1.6.5]).

The Hochschild homology of A and the different variants of cyclic homology are obtained from the mixed Hochschild complex and they are denoted by $\mathrm{HH}_*(A)$, $\mathrm{HC}_*^-(A)$, $\mathrm{HC}_*^{\mathrm{per}}(A)$, $\mathrm{HC}_*(A)$ respectively. The corresponding complexes are denoted by $C(A)$, $CC^-(A)$, $CC^{\mathrm{per}}(A)$ and $CC(A)$. Below we will also use the *reduced* versions of these complexes (and their homology) which are obtained by replacing $C(A)$ by $C(A)/C(l)$.

If $A \in \mathrm{PCAlgc}(l)$ then we define its Hochschild/cyclic homologies through the pseudo-compact version of the Hochschild mixed complex which amounts to taking the *product total complex* of (7.1). It is explained in Appendix B why this is a sensible definition.

Remark 7.1.1. For a pseudo-compact ring the pseudo-compact Hochschild homology is very different from the ordinary Hochschild homology. The following example was pointed out long ago to the author by Amnon Yekutieli.

Let $A = \mathbb{C}[[t]]$. Then the Hochschild homology of A is equal to $\mathrm{Tor}_*^{A^e}(A, A)$. Put $Q = \mathbb{C}((t))$. Then we have

$$\mathrm{Tor}_*^{A^e}(A, A) \otimes_A Q = \mathrm{Tor}_*^{Q^e}(Q, Q)$$

Now Q is simply a field of (very) infinite transcendence degree over \mathbb{C} . By a suitable version of the HKR theorem we obtain

$$\mathrm{HH}_*(A) \otimes_A Q = \bigoplus_i \bigwedge^i \Omega_{Q/\mathbb{C}}$$

and thus $\mathrm{HH}_i(A) \neq 0$ for all i . On the other the Hochschild homology of A as pseudo-compact ring is concentrated in degrees 0, 1 and is given by the continuous version of the HKR theorem.

7.2. X -complexes. We now recall the X -complex formalism as introduced in [6, 29]. By definition an X -complex is a quadruple $\mathcal{U} = (U, V, \partial_0, \partial_1)$ where U, V are DG-vector spaces. $\partial_0 : U \rightarrow V$, $\partial_1 : V \rightarrow U$ are graded maps (of degree zero) commuting with the differentials and $\partial_0 \partial_1 = 0$, $\partial_1 \partial_0 = 0$. An X -complex \mathcal{U} has an associated mixed complex $(M(\mathcal{U}), b, B)$ given by

$$M(\mathcal{U}) = \Sigma V \oplus U$$

$$b = \begin{pmatrix} -d & 0 \\ \partial_1 & d \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \partial_0 \\ 0 & 0 \end{pmatrix}$$

The homological invariants of an X -complex are those of the associated mixed complex.

One may associate an X -complex to a unital l -DG-algebra A . Let⁵

$$(7.2) \quad \Omega_l^1 A = \ker(A \otimes_l A \xrightarrow{a \otimes b \mapsto ab} A)$$

$\Omega_l^1 A$ is the target for the universal l -derivation $D : A \rightarrow \Omega_l^1 A$ which sends a to $Da \stackrel{\text{def}}{=} a \otimes 1 - 1 \otimes a$ (here the assumption that A is unital is used).

For an A -bimodule M put $M_{\natural} = A \otimes_{A^e} M = M/[A, M]$. We define $\partial_0 : A_l \rightarrow (\Omega_l^1 A)_{\natural}$ by $\partial_0(\bar{a}) = \overline{Da}$ and $\partial_1 : (\Omega_l^1 A)_{\natural} \rightarrow A_l$ by $\partial_1(\overline{aDb}) = \overline{[a, b]}$.

Both ∂_0 and ∂_1 commute with the differentials inherited from the DG-structure on A . Furthermore it is easy to see that $\partial_0 \partial_1 = 0$, $\partial_1 \partial_0 = 0$. The X -complex $X(A)$ of A is defined as $(A_l, (\Omega_l^1 A)_{\natural}, \partial_0, \partial_1)$.

There is a canonical morphism of mixed complexes

$$(7.3) \quad \sigma : C(A) \rightarrow M(X(A))$$

defined as follows

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (A^{\otimes_l 4})_l & \xrightarrow{\partial} & (A^{\otimes_l 3})_l & \xrightarrow{\partial} & (A^{\otimes_l 2})_l \xrightarrow{\partial} A_l \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \begin{smallmatrix} a \otimes b \\ \downarrow \\ aDb \end{smallmatrix} & \parallel \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & (\Omega_l^1 A)_{\natural} \xrightarrow{\partial_1} A_l \longrightarrow 0 \end{array}$$

The following is well known.

Proposition 7.2.1. [6] *Assume that A is a cofibrant [24, §1.3] l -DG-algebra. Then σ is a quasi-isomorphism of mixed complexes.*

⁵It is unfortunate that the symbol Ω is used both for differentials and for the cobar construction....

Proof. We may assume that $A = T_l V$ where V is equipped with an ascending filtration $0 = F_0 V \subset F_1 V \subset \dots$ such that $d(F_{i+1} V) \subset T_l F_i V$.

We have a short projective resolution of A as a bimodule

$$(7.4) \quad 0 \rightarrow \Omega_l^1 A \xrightarrow{\partial_1} A \otimes_l A \rightarrow A \rightarrow 0$$

and from the fact that A is cofibrant one deduces that $\Omega_l^1 A$ is a cofibrant A -bimodule. For further reference we note that here ∂_1 is given by

$$\partial_1(aDb) = ab \otimes 1 - a \otimes b$$

Comparing this with the usual resolution given by the shifted bar complex we get a morphism between these resolutions

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{\otimes_l 4} & \longrightarrow & A^{\otimes_l 3} & \longrightarrow & A^{\otimes_l 2} \longrightarrow A \longrightarrow 0 \\ & & \downarrow & & \downarrow \begin{smallmatrix} a \otimes b \otimes c \\ \downarrow \\ a(Db)c \end{smallmatrix} & & \parallel \\ \dots & \longrightarrow & 0 & \longrightarrow & \Omega_l^1 A & \longrightarrow & A^{\otimes_l 2} \longrightarrow A \longrightarrow 0 \end{array}$$

Applying $A \otimes_{A^e} -$ we get a morphism of double complexes which is a quasi-isomorphism on the row level (with the A -differential oriented vertically). We cannot immediately conclude that the sum total complex is acyclic. However putting a suitable filtration on $T_l V$ we may reduce to the case that the differential on A is zero and then it works. \square

The X -complex formalism generalizes without difficulty to the case that $A \in \text{PCAlg}(l)$. Since now the Hochschild complex is a product total complex the analogue of Proposition 7.2.1 is valid under the weaker assumption that A is a tensor algebra when forgetting the differential. This is explained by the fact that the latter objects are precisely the cofibrant objects in $\text{PCAlg}(l)$ (see §A.4).

8. CALABI-YAU ALGEBRAS

We recall some definitions for a k -DG-algebra A .

Definition 8.1. A is *homologically smooth* if A is a perfect A^e -bimodule.

For $M \in D(A^e)$ put $M^D = \text{RHom}_{A^e}(M, A^e)$. The functor $(-)^D$ defines an auto duality on the category of perfect A^e -modules.

Definition 8.2. A is *Calabi-Yau of dimension d* if A is homologically smooth and there exists an isomorphism $\eta : A^D \rightarrow \Sigma^{-d} A$ in the derived category of A^e -modules.

This is a version of Ginzburg's definition of a Calabi-Yau algebra. See [13, Definition 3.2.3]. Ginzburg assumes in addition that $\eta^D = \eta$ but it turns out this is automatic (see Appendix C).

Assume that A is a DG-algebra and M, N are A -DG-bimodules with M perfect. Then in $D(k)$ we have

$$(8.1) \quad \text{RHom}_{A^e}(M^D, N) \cong M \overset{L}{\otimes}_{A^e} N$$

Indeed to prove this we may assume $M = A \otimes A$ and then it is obvious (see also [21, Remark 8.2.4]). We say that $\xi \in H_d(M \overset{L}{\otimes}_{A^e} N)$ is *non degenerate* if the corresponding map $\xi^+ : M^D \rightarrow \Sigma^{-d} N$ is an isomorphism (see Appendix C for more details).

It follows that the isomorphism η in Definition 8.2 defines a non-degenerate element of $\mathrm{HH}_d(A) = H_d(A \overset{L}{\otimes}_{A^e} A)$ and conversely any such non-degenerate element of $\mathrm{HH}_d(A)$ defines an isomorphism $A^D \rightarrow \Sigma^{-d}A$.

Associated to A there is the Connes long exact sequence for cyclic homology.

$$(8.2) \quad \cdots \rightarrow \mathrm{HH}_{d+1}(A) \xrightarrow{I} \mathrm{HC}_{d+1}(A) \xrightarrow{S} \mathrm{HC}_{d-1}(A) \xrightarrow{B} \mathrm{HH}_d(A) \xrightarrow{I} \mathrm{HC}_d(A) \xrightarrow{S} \mathrm{HC}_{d-2}(A) \xrightarrow{B} \mathrm{HH}_{d-1}(A) \rightarrow \cdots$$

The following strengthening of the notion of Calabi-Yau was suggested by Bernhard Keller. It is similar to a dual notion defined by Kontsevich and Soibelman [21, §10.2].

Definition 8.3. A is *exact Calabi-Yau of dimension d* if A is homologically smooth and $\mathrm{HC}_{d-1}(A)$ contains an element η such that $B\eta$ is non-degenerate.

These definitions and observations extend without difficulty to the case that $A \in \mathrm{PCAlg}(l)$. Following [20] we say that an object in $D(A)$ is *strictly perfect* if it is contained in the smallest thick subcategory of $D(A)$ containing A . We say that A is (topologically) homologically smooth if A is strictly perfect in $D(A^e)$.

Thanks to the model structure on $\mathrm{PCDGM}(A^e)$ introduced in §6 we may define $\mathrm{RHom}_{A^e}(A, A^e)$ and so Definition 8.2 is meaningful. Furthermore the derivation of (8.1) is still valid so the concept of a non-degenerate element in $\mathrm{HH}_d(A)$ (temporarily defined as $H_d(A \overset{L}{\otimes}_{A^e} A)$) makes sense.

Thanks to Proposition B.1 $H_*(A \overset{L}{\otimes}_{A^e} A)$ may be computed using the pseudo-compact Hochschild complex which is itself a mixed complex. Hence we have the corresponding long exact sequence for cyclic homology and so Definition 8.3 makes sense as well.

9. CYCLIC HOMOLOGY FOR PSEUDO-COMPACT (NON DG-)CALABI-YAU ALGEBRAS

The following is a pseudo-compact version of Goodwillie's theorem [15, Thm III.5.1].

Theorem 9.1. *Assume that k has characteristic zero and let $A \in \mathrm{Alg}(l)$ be concentrated in degree ≤ 0 . Then $\mathrm{HC}_*^{\mathrm{per}, \mathrm{red}}(A) = 0$.*

Proof of Theorem 9.1. Let $I = \mathrm{rad} A$. We must prove that

$$\mathrm{HC}^{\mathrm{per}}(A) \rightarrow \mathrm{HC}^{\mathrm{per}}(A/I)$$

is an isomorphism. As the I -adic filtration on A is dual to the coradical filtration on $\mathbb{D}A$, all A/I^n are pseudo-compact and furthermore $A = \mathrm{proj} \lim_n A/I^n$.

The functor $\mathrm{proj} \lim$ commutes with (completed) tensor product so we also have

$$\mathrm{C}(A) = \mathrm{proj} \lim_n \mathrm{C}(A/I^n)$$

Since A is concentrated in degree ≤ 0 the same is true for $\mathrm{C}(A)$ and hence the complex $(\mathrm{C}(A)((u)), b + uB)$ computing periodic cyclic homology (see §7.1) involves only products. Since inverse limits commute with products we get

$$(\mathrm{C}(A)((u)), b + uB) = \mathrm{proj} \lim_n (\mathrm{C}(A/I^n)((u)), b + uB)$$

Now by Goodwillie's theorem [15, Thm II.5.1] for nilpotent extensions⁶ the morphism of complexes $(C(A/I^n)((u)), b + uB) \rightarrow (C(A/I)((u)), b + uB)$ is a quasi-isomorphism. As filtered inverse limits are exact in the category of pseudo-compact vector spaces we deduce that $\text{proj lim}_n (C(A/I^n)((u)), b + uB) \rightarrow (C(A/I)((u)), b + uB)$ is a quasi-isomorphism as well. \square

Corollary 9.2. *Assume that k has characteristic zero and let $A \in \text{PCAlg}(l)$ be concentrated in degree zero. Assume in addition that A is d -Calabi-Yau. Then $\text{HC}_i^{\text{red}}(A) = 0$ for $i \geq d$.*

Proof. The algebra $\text{Ext}_A^*(l, l)$ is finite dimensional and symmetric with an invariant form of degree d (see [17, Lemma 3.4]). In particular $\text{Ext}_A^i(l, l) = 0$ for $i > d$.

Consider a minimal free resolution of A as A^e -bimodule

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow A^e \rightarrow A \rightarrow 0$$

Tensoring on the right with l and applying $\text{Hom}_A(-, l)$ we see by Nakayama's lemma that this resolution has length d . Hence $\text{HH}_i(A) = 0$ for $i > d$.

By the reduced version of (8.2) we deduce for $i \geq d$

$$(9.1) \quad \text{HC}_i^{\text{red}}(A) = \text{HC}_{i+2}^{\text{red}}(A) = \cdots$$

Now

$$(9.2) \quad \text{HC}_i^{\text{per, red}}(A) = \text{proj lim}_n \text{HC}_{i+2n}^{\text{red}}(A)$$

The proof of this fact is even easier than in the non-pseudo-compact case (see [25, Prop. 5.1.9]) because inverse limits are exact for pseudo-compact vector spaces.

Combining (9.1)(9.2) with Theorem 9.1 proves what we want. \square

Corollary 9.3. *Assume that k has characteristic zero and let $A \in \text{PCAlg}(l)$ be concentrated in degree zero. Then A is d -Calabi-Yau if and only if it is exact d -Calabi-Yau.*

Proof. Assume that A is d -Calabi-Yau. It follows from Corollary 9.2 that $\text{HC}_d^{\text{red}}(A) = 0$. It now follows from the analogue of (8.2) for reduced homology that any $\eta \in \text{HH}_d(A)$ lifts to an element of $\text{HC}_{d-1}^{\text{red}}(A)$ and hence to an element of $\text{HC}_{d-1}(A)$. This proves what we want. \square

10. DEFORMED DG-PREPROJECTIVE ALGEBRAS

In this section we assume that k has characteristic zero. We will show that pseudo-compact exact Calabi-Yau DG-algebras are obtained from superpotentials. Combining this with Corollary 9.3 it then follows that pseudo-compact (non DG-) Calabi-Yau algebras are derived from superpotentials (and vice versa).

We will give complete proofs but our arguments are certainly heavily inspired by [13, 21, 23]. More precise references will be given below.

⁶The proof of this result extends without difficulty to pseudo-compact DG-algebras concentrated in degrees ≤ 0

10.1. A reminder on non-commutative symplectic geometry. We introduce some notions from [5, 22]. See also [3, 14, 32, 33]. Let A be a graded algebra over $\mathbb{C}(l^e)$. Put

$$\mathrm{DR}_l(A) = T_A(\Omega_l^1 A) / [T_A(\Omega_l^1 A), T_A(\Omega_l^1 A)]$$

Here $T_A(\Omega_l^1 A)$ and $\mathrm{DR}_l(A)$ are considered as bicomplexes. The grading by “form degree” is denoted by $\| - \|$ and the grading derived from the A -grading is denoted by $|-|$.

If ω, ω' are differential forms then their commutator is defined as

$$[\omega, \omega'] = \omega\omega' - (-1)^{\|\omega'\| \|\omega''\| + |\omega'| |\omega''|} \omega'\omega$$

The operator D extends to a derivation $T_A(\Omega_l^1 A) \rightarrow T_A(\Omega_l^1 A)$ homogeneous for $|-|$ and of degree one for $\| - \|$. It is easy to see that D descends to a linear operator on $\mathrm{DR}_l(A)$.

Put $\mathbb{D}\mathrm{er}_l(A) = \mathrm{Der}_{A/l}(A, A \otimes A)$. The elements of $\mathbb{D}\mathrm{er}_l(A) = \mathrm{Der}_{A/l}(A, A \otimes A)$ are referred to as *double derivations*.

Let $\delta \in \mathbb{D}\mathrm{er}_l(A)$. Then the *contraction* i_δ with δ defines a double derivation of degree $(-1, |\delta|)$ on $T_A(\Omega_l^1 A)$ with respect the bigrading $(\| - \|, |-|)$. More precisely

$$i_\delta : \Omega_l^1 A \rightarrow \Omega_l^1 A \otimes \Omega_l^1 A$$

is defined as follows: for $a \in A$ one has

$$\begin{aligned} i_\delta(a) &= 0 \\ i_\delta(Da) &= \delta(a) \end{aligned}$$

Following [5] we put

$$(10.1) \quad \iota_\delta(\omega) = (-1)^{|i_\delta(\omega)'| |i_\delta(\omega)''|} i_\delta(\omega)'' i_\delta(\omega)'$$

Definition 10.1.1. [5] An element $\omega \in \mathrm{DR}_l(A)$ with $\|\omega\| = 2$ which is closed for D is *bisymplectic* if the map of A -bimodules

$$\mathbb{D}\mathrm{er}_l A \rightarrow \Omega_l^1 A : \delta \mapsto \iota_\delta \omega$$

is an isomorphism.

Assume that $\omega \in \mathrm{DR}_l(A)$ is bisymplectic. Following [5] we define the *Hamiltonian vector field* $H_a \in \mathbb{D}\mathrm{er}_l(A)$ corresponding to $a \in A$ via

$$(10.2) \quad \iota_{H_a} \omega = Da$$

and we put

$$(10.3) \quad \llbracket a, b \rrbracket_\omega = H_a(b) \in A \otimes A$$

and

$$\{a, b\}_\omega = \llbracket a, b \rrbracket'_\omega \llbracket a, b \rrbracket''_\omega$$

For the degree of the operations involved one finds by (10.2)

$$|H_a| + |\omega| = |a|$$

and hence by (10.3)

$$|\llbracket a, b \rrbracket_\omega| = |a| + |b| - |\omega|$$

In other words $\llbracket -, - \rrbracket$ has degree $-|\omega|$.

It is shown in [32, App. A] that

$$\llbracket -, - \rrbracket_\omega : A \otimes A \rightarrow A \otimes A$$

is a so-called (graded) “*double Poisson bracket*”. We will not precisely define this notion but we note that it implies that $\{-, -\}_\omega$ descends to a Lie algebra structure on $A/[A, A]$ (this is the Kontsevich bracket, see [22]) and furthermore that it defines an action of $A/[A, A]$ on A by derivations. For use below let us give the precise sign involved in exchanging the arguments of a double Poisson bracket (see [32, §2.7]).

$$(10.4) \quad \llbracket a, b \rrbracket_\omega = -(-1)^{(|a| - |\omega|)(|b| - |\omega|)} (-1)^{|\llbracket a, b \rrbracket'_\omega|} \llbracket a, b \rrbracket''_\omega \otimes \llbracket b, a \rrbracket'_\omega$$

Now assume that $V \in \text{PC}(l^e)$ is finite dimensional and let $\eta \in (V \otimes_l V)_l$ be a non-degenerate element of degree t in the sense that $\eta^+ : V^D \rightarrow \Sigma^t V$ is an isomorphism (see Appendix C), which is furthermore anti-symmetric for β (see §5). Then it is easy to see that

$$\omega_\eta \stackrel{\text{def}}{=} \frac{1}{2} D\eta' D\eta''$$

defines a bisymplectic form of degree t on $A = T_l V$. We note the following.

Lemma 10.1.2. *For all $a \in T_l V$ one has*

$$\{a, \sigma' \eta \sigma''\}_{\omega_\eta} = 0$$

where $\sigma' \eta \sigma'' \in (V \otimes_l V)^l$ (see (5.1)) is viewed as an element of $T_l V$.

Proof.

$$\begin{aligned} \llbracket a, \sigma' \eta \sigma'' \rrbracket_{\omega_\eta} &= \llbracket a, \sigma' \eta \sigma'' \rrbracket'_{\omega_\eta} \otimes \llbracket a, \sigma' \eta \sigma'' \rrbracket''_{\omega_\eta} \\ &= (-1)^{(|a|+t)(|\eta|+t)} (-1)^{|\llbracket a, \sigma' \eta \sigma'' \rrbracket'_{\omega_\eta}|} \llbracket a, \sigma' \eta \sigma'' \rrbracket''_{\omega_\eta} \otimes \llbracket \sigma' \eta \sigma'', a \rrbracket'_{\omega_\eta} \otimes \llbracket \sigma' \eta \sigma'', a \rrbracket''_{\omega_\eta} \end{aligned}$$

(see (10.4) for the signs which are not important here). Thus

$$(10.5) \quad \{a, \sigma' \eta \sigma''\}_{\omega_\eta} = \pm \iota_{H_{\sigma' \eta \sigma''}} Da$$

To continue we make the following claim

$$H_{\sigma' \eta \sigma''} = -\Delta$$

where Δ is the canonical double l -derivation which sends f to $[f, \sigma' \otimes \sigma''] = f \sigma' \otimes \sigma'' - \sigma' \otimes \sigma'' f$.

To prove this claim we use the fact that the defining equation for $H_{\sigma' \eta \sigma''}$ is

$$\iota_{H_{\sigma' \eta \sigma''}} \omega_\eta = D(\sigma' \eta \sigma'')$$

We compute

$$\begin{aligned} 2\iota_\Delta \omega &= (-1)^{|\eta'| |\eta''|} \sigma'' (D\eta'') \eta' \sigma' - \sigma'' \eta' (D\eta'') \sigma' - \sigma'' (D\eta') \eta'' \sigma' + (-1)^{|\eta'| |\eta''|} \sigma'' \eta'' (D\eta') \sigma' \\ &= -\sigma'' (D\eta') \eta'' \sigma' - \sigma'' \eta' (D\eta'') \sigma' - \sigma'' (D\eta') \eta'' \sigma' - \sigma'' \eta' (D\eta'') \sigma' \\ &= -2\sigma'' D(\eta' \eta'') \sigma' \\ &= -2D(\sigma' \eta \sigma'') \end{aligned}$$

where in the second line we have used anti-symmetry of η and in the last line the symmetry of σ . We get indeed $H_{\sigma' \eta \sigma''} = -\Delta$. Plugging this into (10.5) we find

$$\begin{aligned} \{a, \sigma' \eta \sigma''\}_{\omega_\eta} &= \pm \iota_\Delta Da \\ &= \pm (\sigma'' a \sigma' - \sigma'' a \sigma') \\ &= 0 \quad \square \end{aligned}$$

10.2. Deformed DG-preprojective algebras. We now discuss a class of DG-algebras introduced by Ginzburg in [13, §3.6] and (somewhat implicitly) by Lazaroiu in [23, §5.3]. We introduce things in the pseudo-compact context but of course the definitions works just as well in the non-topological case.

Suppose we have the data (V_c, η, w) where

- (1) $V_c \in \text{PC}(l^e)$ is finite dimensional.
- (2) $\eta \in (V \otimes_l V)_l$ is non-degenerate and anti-symmetric under β .
- (3) $w \in T_l V / [T_l V, T_l V]$ satisfies $|w| = |\eta| + 1$ and $\{w, w\}_{\omega_\eta} = 0$.

The *deformed DG-preprojective algebra* $\Pi(V_c, \eta, w)$ associated to this data is the augmented pseudo-compact l -DG-algebra $T_l(V + z l)$ where z is l -central and the differential is given by

$$\begin{aligned} dz &= \sigma' \eta \sigma'' \\ df &= \{w, f\}_{\omega_\eta} \quad \text{for } f \in T_l V_c \end{aligned}$$

It is easy to see that this defines an honest DG-algebra. Indeed to verify $d^2 = 0$ we have to check $d^2 z = 0$ and $d^2 f = 0$ for $f \in V_c$. We find

$$d^2 z = d(\sigma' \eta \sigma'') = \{w, \sigma' \eta \sigma''\}_{\omega_\eta} = 0$$

using Lemma 10.1.2 and

$$(10.6) \quad d^2 f = \{w, \{w, f\}_{\omega_\eta}\}_{\omega_\eta} = \frac{1}{2} \{\{w, w\}_{\omega_\eta}, f\}_{\omega_\eta} = 0$$

Remark 10.2.1. The element w in the definition of a deformed DG-preprojective algebra is commonly called a *superpotential*.

The reason for introducing deformed DG-preprojective algebras is the following.

Theorem 10.2.2. *Assume that $A \in \text{Alg}(l)$ is concentrated in degrees ≤ 0 . Then the following are equivalent.*

- (1) A is exact d -Calabi-Yau.
- (2) A is weakly equivalent to an algebra of the form $\Pi(V_c, \eta, w)$ with V_c concentrated in degrees $[-d+2, 0]$, $|\eta| = -d+2$, $|w| = -d+3$ and w contains only cubic terms and higher.

A related result has been proved by Ginzburg in the non-DG case [13, Thm 3.6.4]. We will prove Theorem 10.2.2 in §11.

Remark 10.2.3. The actual choice of the trace $\text{Tr} : l \rightarrow k$ is irrelevant for the definition of $\Pi(V_c, \eta, \omega)$. Indeed different traces lead to σ 's differing by a central element of l which can be absorbed into z .

Remark 10.2.4. The hypothesis in Theorem 10.2.2 that A is concentrated in degrees ≤ 0 may be inconvenient in some cases. We may relax it as follows. We say that an l -bimodule W has *no oriented cycles of strictly positive degree* if $((T_l W)_l)_{>0} = 0$. One may prove a variant of Theorem 10.2.2 under the hypothesis that A (as l -bimodule) has no oriented cycles of strictly positive degree. The condition that V_c lives in degrees $[-d+2, 0]$ should then also be replaced by the condition that V_c has no oriented cycles of strictly positive degree.

10.3. The quiver case. In this section we discuss quivers. *We implicitly assume that all path algebras are completed at path length.*

We assume that k is algebraically closed (still of characteristic zero) and $l = \sum_{i=1}^n ke_i$ for central orthogonal idempotents $(e_i)_i$. If we are given (V_c, η) where V_c is a finite dimensional graded l -bimodule and η is a non-degenerate anti-symmetric element in $(V_c \otimes_l V_c)_l$ then we may write

$$\eta = \sum_{ijt} [x_{ij}^t, x_{ji}^{t*}] \quad \text{mod } [l, -]$$

where $\{x_{ij}^t, x_{ji}^{t*}\}$ represents a homogeneous basis for $e_i V_c e_j$ with $|x_{ij}^t| \geq |\eta|/2$. We assume that the x_{ij}^t, x_{ji}^{t*} are all distinct except for the case $i = j$ where we assume $x_{ii}^t = x_{ii}^{t*}$ if $|x_{ii}^t| = |x_{ii}^{t*}| = |\eta|/2$ is odd.

As usual the x_{ij}^t, x_{ji}^{t*} may be regarded as arrows in a quiver \tilde{Q} with vertices $\{1, \dots, n\}$ and it is easy to check that in that case $\{-, -\}_{\omega_\eta}$ is precisely the *necklace bracket* on $T_l V_c = k\tilde{Q}$ [3, 14, 22]. For a pictorial presentation of the necklace bracket as an operation

$$k\tilde{Q}/[k\tilde{Q}, k\tilde{Q}] \times k\tilde{Q} \rightarrow k\tilde{Q}$$

see [32, Prop. 6.4.1].

If we let Q be the graded quiver with arrows x_{ij}^t of degree $\geq |\eta|/2$ then \tilde{Q} is obtained from Q by the following procedure.

- (1) For every non-loop $a \in Q$ we adjoin an arrow a^* in the opposite direction such that $|a^*| + |a| = |\eta|$.
- (2) We do the same for loops, *except* for a loop a of odd degree such that $|a| = |\eta|/2$. In that case we put $a^* = a$ (thus we do not adjoin an extra arrow).

Taking into account Remark 10.2.3 we may assume that $\sigma = \sum_i e_i \otimes e_i$. This leads to the following construction. We start with the data (Q, d, w) consisting of (1) an integer d , (2) a finite graded quiver Q with arrows of degree $\geq (2-d)/2$, (3) an element $w \in k\tilde{Q}/[k\tilde{Q}, \tilde{Q}]$ (with \tilde{Q} as above with $|\eta| = 2-d$) satisfying $\{w, w\} = 0$ for the necklace bracket on $k\tilde{Q}/[k\tilde{Q}, k\tilde{Q}]$.

Let \bar{Q} be obtained from \tilde{Q} by adjoining at every vertex i a loop z_i of degree $1-d$. Put $z = \sum_i z_i$. Then $\Pi(Q, d, w)$ is the DG-algebra $(k\bar{Q}, d)$ with differential

$$(10.7) \quad dz = \sum_{a \in Q} [a, a^*]$$

$$(10.8) \quad df = \{w, f\} \quad \text{for } f \in k\tilde{Q}$$

By the above discussion we have

$$\Pi(Q, d, w) \cong \Pi(V_c, \eta, w)$$

where

$$\eta = \sum_{a \in Q} [a, a^*]$$

and V_c is the l -bimodule corresponding to the quiver \tilde{Q} .

Remark 10.3.1. The condition (10.8) may be rewritten as

$$(10.9) \quad \begin{aligned} da &= (-1)^{(|a|+1)|a^*|} \frac{\circ \partial w}{\partial a^*} \\ da^* &= -(-1)^{|a|} \frac{\circ \partial w}{\partial a} \end{aligned}$$

for $a \in Q$ where $\circ \partial / \partial x$ for is the *circular derivative*

$$\frac{\circ \partial w}{\partial x} = \sum_{w=uxv} (-1)^{|u|(|x|+|v|)} vu$$

Remark 10.3.2. Since we are assuming that k is algebraically closed we can in fact always reduce to the quiver case up to Morita equivalence. Indeed we have $l = \bigoplus_i l_i$ for $l_i = M_{p_i}(k)$. Let e be an idempotent in l such that ele is isomorphic to the center $Z(l)$ of l . Clearly $\Pi(V_c, \eta, w)$ is Morita equivalent to $e\Pi(V_c, \eta, w)e$.

Put $V'_c = eV_c e$. Then $(V_c \otimes_l V_c)_l \cong (V'_c \otimes_{Z(l)} V'_c)_{Z(l)}$. So η corresponds to an element η' of $(V'_c \otimes_{Z(l)} V'_c)_{Z(l)}$. Put $w' = ewe$. One verifies easily that $\{w', w'\}_{\omega_{\eta'}} = 0$ and

$$e\Pi(V_c, \eta, w)e \cong \Pi(V'_c, \eta', w')$$

11. PROOF OF THEOREM 10.2.2

As the title of this section indicates we will prove Theorem 10.2.2. The proof consists of a number of steps. In §11.1 we will translate the non-degeneracy condition for Hochschild cycles into a more tractable form. In §11.2 we obtain a first classification of exact Calabi-Yau algebras. Finally in §11.3 we complete the proof.

11.1. Non-degeneracy in the cofibrant case. In this section we assume that $A = (T_l V, d)$ is a cofibrant object in $\text{PCAlg}(l)$ (see §A.4). We write the differential as $d = d_1 + d_2 + \dots$ as in §A.5.

By the pseudo-compact analogue of Proposition 7.2.1 (see the end of §7.2) we have $C(A) \cong \text{cone}((\Omega_A^1)_\natural \xrightarrow{\partial_1} A_l)$. An element ξ of $\text{HH}_d(A)$ is now represented by a pair (ω, a) where $\omega \in (\Omega_l^1 A)_\natural$, $a \in A_l$, $d\omega = 0$, $\partial_1 \omega = da$, $|\omega| = d - 1$, $|a| = d$. We would like to know when ξ is non-degenerate. We answer this question in a typical case in Lemma 11.1.2 below.

We have a morphism of X -complexes

$$\begin{array}{ccccc} A_l & \xrightarrow{\partial_0} & (\Omega_l^1 A)_\natural & \xrightarrow{\partial_1} & A_l \\ \downarrow & & \downarrow \text{res} & & \downarrow \\ l_l & \xrightarrow{0} & V_l & \xrightarrow{0} & l_l \end{array}$$

The outermost maps send $f \in A$ to its image in \bar{f} in l . The middle map sends fDv for $v \in V$ to $\bar{f}v$. We obtain a corresponding map

$$(11.1) \quad C(A) \rightarrow ((l \oplus \Sigma V)_l, d_1)$$

of complexes.

Remark 11.1.1. It is easy to show that that (11.1) can be described intrinsically as being obtained from the the standard map

$$C(A) = A \overset{L}{\otimes}_{A^e} A \rightarrow (l \overset{L}{\otimes}_A l)_l$$

We present it in the above explicit way since that is how we will use it.

Lemma 11.1.2. *Assume V is of the following form*

- (1) $d_1 : V \rightarrow V$ is zero.
- (2) $V = V_c \oplus lz$ with z an l -central element of degree $-d + 1$ and V_c finite dimensional.
- (3) V_c is concentrated in degrees $[-d + 2, 0]$.
- (4) $dz = \sigma' \eta \sigma'' \bmod V^{\otimes 3}$ with $\eta \in (V_c \otimes_l V_c)_l$ being a non-degenerate element of degree $-d + 2$.

Then (ω, a) is non degenerate if and only if $\text{res } \omega = uz^\dagger$ for u a central unit in l (see (5.1) for the notation $(-)^{\dagger}$).

We use the following preparatory lemma.

Lemma 11.1.3. *Let M, N be objects in $\text{PCDGM}od(A^e)$ which are of the form $(A \otimes_l M_0 \otimes_l A, d)$, $(A \otimes_l N_0 \otimes_l A, d)$, such that $l^e \otimes_{A^e} M = M_0$, $l^e \otimes_{A^e} N = N_0$ are finite dimensional and have zero differential. Let $\xi \in H^u(M \otimes_{A^e} N)$. The following are equivalent*

- (1) ξ is non-degenerate.
- (2) The image $\bar{\xi}$ of ξ in $H^u(M_0 \otimes_{l^e} N_0)$ is non-degenerate.

Proof. We prove (1) \Rightarrow (2). If ξ is non-degenerate then it produces an isomorphism in $D(A^e)$

$$(11.2) \quad \xi^+ : \text{Hom}_{A^e}(M, A \otimes A) \rightarrow N$$

Both sides are cofibrant, left and right tensoring with l gives us an isomorphism in $D(l^e)$

$$(11.3) \quad \text{Hom}_{l^e}(M_0, l \otimes l) \rightarrow N_0$$

which is easily shown to equal to $\bar{\xi}^+$. Hence $\bar{\xi}$ is non-degenerate.

Now we prove (2) \Rightarrow (1). Since $\bar{\xi}$ is non-degenerate and since M_0, N_0 have zero differential we see that $\bar{\xi}$ induces an isomorphism $\text{Hom}_{l^e}(M_0, l \otimes l) \cong N_0$ as l -bimodules. It follows immediately from Nakayama's lemma applied to A^e (without differential) that $\xi^+ : \text{Hom}_{A^e}(M, A \otimes A) \rightarrow N$ is an isomorphism of graded A -modules. As this isomorphism is compatible with the differential it is an isomorphism of DG-bimodules. \square

Proof of Lemma 11.1.2. We will first prove the \Leftarrow direction. Thus if $\xi = (\omega, a) \in \text{HH}_d(A)$ and $\text{res } \omega = uz^\dagger$ for u a central unit in l then we must prove that ξ is non-degenerate. Replacing z by uz we may assume $u = 1$.

We will view ξ as a cohomology class of degree $-d$ in the total complex of the triple complex obtained from tensoring the resolution (7.4) with itself over A^e . This

triple complex looks as follows

$$(11.4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega_l^1 A \otimes_{A^e} (A \otimes_l A) & \xrightarrow{\partial_1 \otimes 1} & (A \otimes_l A) \otimes_{A^e} (A \otimes_l A) & \longrightarrow & 0 \\ & & \uparrow 1 \otimes \partial_1 & & \uparrow 1 \otimes \partial_1 & & \\ 0 & \longrightarrow & \Omega_l^1 A \otimes_{A^e} \Omega_l^1 A & \xrightarrow{\partial_1 \otimes 1} & (A \otimes_l A) \otimes_{A^e} \Omega_l^1 A & \longrightarrow & 0 \\ & & \uparrow 0 & & \uparrow 0 & & \end{array}$$

Or simplified

$$(11.5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (\Omega_l^1 A)_l & \xrightarrow{\partial_1^{\text{hor}}} & (A \otimes_l A)_l & \longrightarrow & 0 \\ & & \uparrow 1 \otimes \partial_1 & & \uparrow \partial_1^{\text{ver}} & & \\ 0 & \longrightarrow & \Omega_l^1 A \otimes_{A^e} \Omega_l^1 A & \xrightarrow{\partial_1 \otimes 1} & (\Omega_l^1 A)_l & \longrightarrow & 0 \\ & & \uparrow 0 & & \uparrow 0 & & \end{array}$$

where the only ambiguous identification we have used is

$$(11.6) \quad (A \otimes_l A) \otimes_{A^e} (A \otimes_l A) \rightarrow (A \otimes_l A)_l : (a \otimes b) \otimes (c \otimes d) \mapsto (-1)^{|d|(|a|+|b|+|c|)} da \otimes bc$$

For further reference we denote this triple complex by $Y(A)$.

ξ is now represented by a sum $\sum_i s\omega_i \otimes s\omega'_i + s\psi + s\psi' + c$ where $\omega_i, \omega'_i \in \Omega_l^1 A$, $\psi, \psi' \in (\Omega_l^1 A)_l$, $c \in (A \otimes_l A)_l$.

From the condition $(d + \partial_1 \otimes 1 + 1 \otimes \partial_1)(\xi) = 0$ we deduce the following conditions in $\Sigma(\Omega_l^1 A)_l$ (viewed respectively as $(A \otimes_l A) \otimes_{A^e} \Sigma\Omega_l^1 A$ and $\Sigma\Omega_l^1 A \otimes_{A^e} (A \otimes_l A)$).

$$\begin{aligned} -sd\psi + \sum_i \partial_1 \omega_i \otimes s\omega'_i &= 0 \\ -sd\psi' + \sum_i (-1)^{|\omega_i|+1} s\omega_i \otimes \partial_1 \omega'_i &= 0 \end{aligned}$$

which in $(\Omega_l^1 A)_l$ becomes

$$(11.7) \quad \begin{aligned} d\psi &= \sum_i (-1)^{|\omega_i|} \partial_1 \omega_i \otimes \omega'_i \\ d\psi' &= \sum_i (-1)^{|\omega_i|+1} \omega_i \otimes \partial_1 \omega'_i \end{aligned}$$

We consider these identities modulo $V^2 \cdot DV + V \cdot DV \cdot V + DV \cdot V^2$ in $\Omega_l^1 A$.

Keeping only the terms that can contribute (using the fact that $|\xi| = d$, and the fact that $d_1 = 0$) we get

$$\begin{aligned}\psi &= Dz^\dagger + \dots \\ \psi' &= Dw + \dots \\ \sum_i \omega_i \otimes \omega'_i &= \sum_j Du_j \otimes Dv_j + \dots\end{aligned}$$

for $w, u_j, v_j \in V$.

Translating (11.7) we get in $(\Omega_l^1 A)_l$.

$$\begin{aligned}D(dz^\dagger)_2 &= \sum_j (-1)^{|u_j|} ((-1)^{|u_j||v_j|} (Dv_j)u_j - u_j(Dv_j)) \\ D(dw)_2 &= \sum_j (-1)^{|u_j|+1} ((Du_j)v_j - (-1)^{|u_j||v_j|} v_j(Du_j))\end{aligned}$$

where $(-)_2$ denotes the quadratic part.

Projecting on $(DV \cdot V)_l \cong (V \otimes_l V)_l$ and $(V \cdot DV)_l \cong (V \otimes_l V)_l$ we get

$$\begin{aligned}(dz^\dagger)'_2 \otimes (dz^\dagger)''_2 &= \sum_j (-1)^{|u_j|} (-1)^{|u_j||v_j|} v_j \otimes u_j \\ (dz^\dagger)'_2 \otimes (dz^\dagger)''_2 &= \sum_j (-1)^{|u_j|+1} u_j \otimes v_j \\ (dw)'_2 \otimes (dw)''_2 &= \sum_j (-1)^{|u_j|+1} u_j \otimes v_j \\ (dw)'_2 \otimes (dw)''_2 &= \sum_j (-1)^{|u_j|} (-1)^{|u_j||v_j|} v_j \otimes u_j\end{aligned}$$

We conclude $dz_2^\dagger = dw_2 \pmod{[l, -]}$ and furthermore

$$\sum_j (-1)^{|u_j|(|v_j|+1)} v_j \otimes u_j = \eta_2 \pmod{[l, -]}$$

We will apply Lemma 11.1.3 to $Y(A)$. We find that $\bar{\xi}$ is of the form

$$sDz \otimes 1 + 1 \otimes sDz + (-1)^{|\eta'|(|\eta''|+1)} sD\eta' \otimes sD\eta''$$

By the non-degeneracy of η , this is clearly a non-degenerate element of $(l \oplus \Sigma V) \otimes (l \oplus \Sigma V)$. Hence we are done.

Now we prove the “easy” direction \Rightarrow . Let $\xi = \sum_i s\omega_i \otimes s\omega'_i + s\psi + s\psi' + c$ in (11.5) represent a non-degenerate element in $\mathrm{HH}_d(A)$. Then

$$\bar{\xi} = \sum_j su_j \otimes sv_j + sw \otimes 1 + 1 \otimes sw'$$

for $u_j, v_j, w, w' \in V$ where we have identified $l^e \otimes_{A^e} \Omega_l^1 A$ with $(DV)_l \cong V_l$. This element must be non-degenerate in $((\Sigma V \oplus l) \otimes (\Sigma V \oplus l))_l$ by Lemma 11.1.3. This can only happen if $w \in V_l$ is a generator for $(V_{-d+1})_l$. Then w must necessarily be of the form uz^\dagger for an invertible element $u \in Z(l)$. \square

11.2. First classification of exact Calabi-Yau algebras.

Theorem 11.2.1. *Assume the characteristic of k is zero. Let $A \in \text{PCAlgc}(l)$. Assume that A is concentrated in degrees ≤ 0 . Then the following are equivalent.*

- (1) *A is exact Calabi-Yau.*
- (2) *There is a weak equivalence $(T_l V, d) \rightarrow A$ as augmented l -DG-algebras with V having the following properties*
 - (a) *$d_1 : V \rightarrow V$ is zero.*
 - (b) *$V = V_c \oplus lz$ with z an l -central element of degree $-d+1$ and V_c finite dimensional.*
 - (c) *V_c is concentrated in degrees $[-d+2, 0]$.*
 - (d) *$dz = \sigma' \eta \sigma''$ with $\eta \in (V_c \otimes_l V_c)_l$ being a non-degenerate and anti-symmetric element under β (cfr. §5.2).*
- (3) *There is a weak equivalence $(T_l V, d) \rightarrow A$ as augmented l -DG-algebras with V as in (2a-2c) and (2d) is replaced by*
 - (d') *$dz = \sigma' \eta \sigma'' = \sigma' \eta_2 \sigma'' + \sigma' \eta_3 \sigma'' + \dots$ such that η is a sum of commutators in $T_l V_c$, $\eta_n \in (V^{\otimes n})_l$ and $\eta_2 \in (V_c \otimes_l V_c)_l$ is non-degenerate.*

Proof. We first prove the direction (1) \Rightarrow (3). We employ Koszul duality (see §A.5). Since A is exact d -Calabi-Yau it is in particular d -Calabi-Yau and hence the algebra $\text{Ext}_A^*(l, l)$ is finite dimensional and symmetric with an invariant form of degree d (see [17, Lemma 3.4]). Thus by Corollary A.5.2 $H^*(A^!)$ is symmetric with an invariant form of degree d . From the construction of $A^!$ (cfr. (A.9)) it follows immediately $\bar{A}^!$ is concentrated in degrees ≥ 1 . Hence the same holds for $H^*(A)$. Thus $H^*(A^!) = l \oplus W_c \oplus lh$ with h being l -central of degree d and W_c being concentrated in degrees $[1, d-1]$. The h element is constructed as the orthogonal to the augmentation.

By Proposition A.5.4 this implies that A has a minimal model $(T_l V, d)$ with $V = V_c \oplus lz$ as in Lemma 11.1.2 (with η in addition being anti-symmetric under β).

By Corollary D.4 we may replace A by $(T_l V, d)$. Assuming now $A = (T_l V, d)$ we note that in characteristic zero the third quadrant double complex

$$\dots \xrightarrow{\partial_1} (A/l)_l \xrightarrow{\partial_0} (\Omega_l^1 A)_\natural \xrightarrow{\partial_1} (A/l)_l \rightarrow (A/l + [A, A])_l \rightarrow 0$$

has exact rows (see [25, Thm 3.1.4] and proof). In other words the reduced cyclic homology of A is equal to $(A/l + [A, A])_l$. The Connes long exact sequence for (reduced) cyclic homology is obtained from the distinguished triangle

$$\Sigma(A/l + [A, A])_l \xrightarrow[B]{\partial_0} \text{cone}((\Omega_l^1 A)_\natural \xrightarrow{\partial_1} (A/l)_l) \xrightarrow{I} (A/l + [A, A])_l \rightarrow$$

Let $\xi \in \text{HC}_{d-1}(A)$ be such that $B\xi$ is non-degenerate. Then ξ corresponds to an element $\bar{\chi}$ of degree $-d+1$ in $(A/l + [A, A])_l$ with $d\bar{\chi} = 0$. In other words

$$(11.8) \quad d\chi = \sum_i [x_i, y_i] \quad \text{mod}[l, -]$$

for $x_i, y_i \in \bar{A}$. We put $\eta' = \sum_i [x_i, y_i]$.

The element $\bar{\chi}$ is sent under ∂_0 to $(D\chi, 0)$. Since $(D\chi, 0)$ is a non-degenerate element of Hochschild homology it follows from Lemma 11.1.2 that $\chi = uz^\dagger + v$ for u an invertible central element in l and $v \in \bar{A}^2$. Put $z' = \sigma' \chi \sigma''$ and $V' = V_c + lz'$. From (11.8) we deduce

$$dz' = \sigma' \eta' \sigma'' = \sigma' \eta'_2 \sigma'' + \sigma' \eta'_3 \sigma'' + \dots$$

such that η' is a sum of commutators in $T_l V_c$, $\eta'_n \in (V_c^{\otimes n})_l$ and $\eta'_2 \in (V_c \otimes_l V_c)_l$. As $T_l V = T_l V'$ we obtain from Corollary A.5.6 that $V' = \Sigma^{-1} \mathbb{D} \overline{A}$ and hence η'_2 is still non-degenerate.

Now we prove (3) \Rightarrow (2). This is a version of [21, Prop. 10.1.2]. We first note that the condition $dz = \sigma' \eta \sigma''$ with η being a sum of commutators in $T_l V_c$ is obviously invariant under isomorphisms $q : (T_l V, d) \rightarrow (T_l V, d')$ of the form $q(v) = v + \text{higher terms}$ with $q(z) = z$ and $q(T_l V_c) \subset T_l V_c$ (this last condition is in fact automatic for degree reasons).

Assume that we have shown that A is weakly equivalent to $(T_l V, d)$ such that $dz = \sigma' \eta \sigma''$ with $\eta_3 = \dots = \eta_{n-1} = 0$. We will construct an isomorphism $q : (T_l V, d) \rightarrow (T_l V, d')$ of augmented pseudo-compact l -DG-algebra of the form $q(v) = v + \beta(v)$ for $\beta(v) \in V_c^{\otimes n-1}$ such that $\beta(z) = 0$ and such that $d'z = \sigma' \eta' \sigma''$ with $\eta'_n = 0$ (and of course $\eta_i = \eta'_i$ for $i \leq n-1$). Repeating this procedure we kill in the limit all the higher order terms of dz .

We have $d' = q \circ d \circ q^{-1}$.

$$\begin{aligned} d'z &= qdz \\ &= \sigma' q(\eta) \sigma'' \\ &= \sigma' \eta_2 \sigma'' + \sigma' \eta_n \sigma'' + \sigma' \beta(\eta'_2) \eta''_2 \sigma'' + \sigma' \eta'_2 \beta(\eta''_2) \sigma'' + \dots \end{aligned}$$

Thus we must solve the following equation in $(V_c^{\otimes n})_l$

$$\eta_n + \beta(\eta'_2) \eta''_2 + \eta'_2 \beta(\eta''_2) = 0$$

for an l -bimodule map $\beta : V_c \rightarrow V_c^{\otimes n-1}$. This can be rewritten as

$$\begin{aligned} (11.9) \quad 0 &= \eta_n - (-1)^{|\eta'_2| |\eta''_2|} \beta(\eta''_2) \eta'_2 + \eta'_2 \beta(\eta''_2) \\ &= \eta_n - [\eta'_2, \beta(\eta''_2)] \end{aligned}$$

As (11.9) is a linear algebra problem we may without loss of generality assume that k is algebraically closed. Thus $l = \bigoplus_i l_i$ for $l_i = M_{p_i}(k)$.

It is furthermore easy to see that (11.9) is invariant under Morita equivalence. Therefore we may replace l by its center and so we are reduced to $l = \sum_{i=1}^m k e_i$ for central orthogonal idempotents $(e_i)_i$. I.e. the “quiver case”.

As in §10.3 we may bring η_2 in the following form

$$\eta_2 = \sum_{a \in Q} [a, a^*]$$

where Q is a suitable graded quiver with vertices $\{1, \dots, m\}$. Now one verifies that η_n , being a sum of commutators can be written as

$$\eta_n = \sum_{a \in Q} [a, \eta_a] + [a^*, \eta_{a^*}] \quad (\text{modulo } [l, -])$$

for certain paths η_a, η_{a^*} of length $n-1$ in \tilde{Q} (the quiver corresponding to the l -bimodule V_c , see §10.3). If $a = a^*$ then we may and we will assume $\eta_a = \eta_{a^*}$. It now suffices to define

$$\begin{aligned} \beta(a) &= -(-1)^{|a| |a^*|} \eta_{a^*} \\ \beta(a^*) &= \eta_a \end{aligned}$$

to obtain the solution to (11.9). Note that if $a = a^*$ then $|a| = |a^*|$ is odd and hence $-(-1)^{|a| |a^*|} = 1$, as it should.

Finally we prove the direction (2) \Rightarrow (1). We consider \bar{z}^\dagger as an element of $\mathrm{HC}_{d-1}^{\mathrm{red}}(A) = H^{-d+1}(A/(l + [A, A]))_l$ as we have indeed $d\bar{z}^\dagger = 0$. Then $\partial_0 \bar{z}^\dagger = (Dz^\dagger, 0)$. Since $\mathrm{res} Dz^\dagger = z^\dagger$ it is now sufficient to invoke Lemma 11.1.2. \square

11.3. Completion of proof. Theorem 10.2.2 now follows from the equivalence of (1) and (2) in Theorem 11.2.1 together with the following lemma.

Lemma 11.3.1. *Assume that k has characteristic zero. Let $A = (T_l V, d)$ be an augmented pseudo-compact l -DG-algebra with the following properties.*

- (1) $V = V_c \oplus lz$ with z an l -central element of degree $-d + 1$ and V_c finite dimensional.
- (2) $dz = \sigma' \eta \sigma''$ with $\eta \in (V_c \otimes_l V_c)_l$ being non-degenerate and anti-symmetric under \mathfrak{B} .
- (3) $T_l V_c$ is stable under d .

Then there is some $w \in T_l V / [T_l V, T_l V]$ with $|w| = |\eta| + 1 = -d + 3$ and $\{w, w\}_{\omega_\eta} = 0$ such that for $f \in T_l V_c$ we have

$$df = \{w, f\}_{\omega_\eta}$$

Hence $A = \Pi(V_c, \eta, w)$. Furthermore if $d_1 : V \rightarrow V$ is zero then w will have only cubic terms and higher.

Proof. This can be deduced from the general machinery of non-commutative symplectic geometry but we will give an explicit proof. Since $d^2 = 0$ we obtain

$$\sigma'((d\eta')\eta'' + (-1)^{|\eta'|}\eta'(d\eta''))\sigma'' = 0$$

and hence

$$(d\eta')\eta'' + (-1)^{|\eta'|}\eta'(d\eta'') = 0 \quad \text{mod } [l, -]$$

This can be rewritten as (everything mod $[l, -]$)

$$\begin{aligned} (-1)^{|\eta'|}\eta'(d\eta'') &= -(d\eta')\eta'' \\ &= (-1)^{|\eta'|\|\eta''\|}(d\eta'')\eta' \end{aligned}$$

where the last identity follows from applying $d \otimes 1$ to $\eta' \otimes \eta'' = -(-1)^{|\eta'|\|\eta''\|}\eta'' \otimes \eta'$. In other words

$$(11.10) \quad \bar{w} \stackrel{\mathrm{def}}{=} (-1)^{|\eta'|+1}\eta'd\eta''$$

is a cyclically symmetric element of $(T_l V_c)_l$ of degree $-d + 3$.

If $\phi \in V_c^D$ then we define

$$\partial_\phi : (T_l V_c)_l \rightarrow T_l V_c : a_1 \otimes \cdots \otimes a_n \rightarrow \phi(a_1)'' a_2 \otimes \cdots \otimes a_n \phi(a_1)'$$

(there is no sign here since $\phi(a_1)'$ is a scalar).

The element $\eta \in (V_c \otimes_l V_c)_l$ of degree $-d + 2$ defines a map

$$\eta^+ : V_c^D \rightarrow V_c : \phi \mapsto (-1)^{|\phi|\|\eta\|}\phi(\eta')''\eta''\phi(\eta')'$$

of degree $-d + 2$ (again the fact that $\phi(\eta')'$ is a scalar makes the sign rather trivial).

Using the definition of \bar{w} we obtain the following identity

$$d(\eta^+(\phi)) = (-1)^{|\phi|\|\eta\|}(-1)^{|\phi|+1}\partial_\phi(\bar{w})$$

(where we use that for non-zero terms we have $|\phi| + |\eta'| = 0$).

Let w be an inverse image of \bar{w} under the cyclic symmetrization map

$$(T_l V_c)_l \rightarrow (T_l V_c)_l : a_1 \otimes \cdots \otimes a_n \rightarrow \sum_i \pm a_i \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}$$

E.g. one possible choice is

$$w = \sum_n \frac{\bar{w}_n}{n}$$

where $\bar{w}_n \in (V_c^{\otimes n})_l$.

If $\phi \in V_c^D$ then we have an associated double l -derivation

$$i_\phi : T_l V_c \rightarrow T_l V_c \otimes T_l V_c$$

which sends v to $\phi(v)$. We get an induced map

$$\iota_\phi : T_l V_c / [T_l V_c, T_l V_c] \rightarrow T_l V_c : \bar{f} \mapsto (-1)^{|i_\phi(f)''| |i_\phi(f)'|} i_\phi(f)'' i_\phi(f)'$$

which sends $a_1 \otimes \cdots \otimes a_n$ to

$$\sum_i \pm \phi(a_i)'' a_{i+1} \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1} \phi(a_i)'$$

It follows that

$$\iota_\phi(w) = \partial_\phi(\bar{w})$$

and thus

$$(11.11) \quad d(\eta^+(\phi)) = (-1)^{|\phi||\eta|} (-1)^{|\phi|+1} \iota_\phi(w)$$

Since η is non-degenerate we have an inverse to η^+

$$\eta^- : V_c \rightarrow V_c^D$$

of degree $d-2$. Applying (11.11) with $\phi = \eta^-(v)$ for $v \in V_c$ yields

$$(11.12) \quad dv = (-1)^{(|v|+d-2)(d-2)+(|v|+d-2)+1} \iota_{\eta^-(v)}(w) = (-1)^{|v|(d+1)+1} \iota_{\eta^-(v)}(w)$$

We have

$$\begin{aligned} v = \eta^+(\phi) &= (-1)^{|\phi||\eta|} \phi(\eta')'' \eta'' \phi(\eta')' \\ &= -(-1)^{|\phi||\eta|} (-1)^{|\eta'| |\eta''|} \phi(\eta'')'' \eta' \phi(\eta'')' \end{aligned}$$

Hence

$$\begin{aligned} 2\iota_\phi \omega_\eta &= \iota_\phi D\eta' D\eta'' \\ &= \phi(\eta')'' (D\eta'') \phi(\eta')' - (-1)^{|\eta'| |\phi|} \phi(\eta'')'' (D\eta') \phi(\eta'')' \\ &= 2(-1)^{|\phi||\eta|} Dv \end{aligned}$$

So ultimately we find

$$\iota_{\eta^-(v)}(\omega_\eta) = (-1)^{(|v|+d-2)(d-2)} Dv = (-1)^{d(|v|+1)} Dv$$

and hence

$$\eta^-(v) = (-1)^{d(|v|+1)} H_v$$

Comparing with (11.12) we find

$$\begin{aligned} dv &= (-1)^{|v|(d+1)+1} (-1)^{d(|v|+1)} \iota_{H_v}(w) \\ &= (-1)^{|v|+d+1} \iota_{H_v}(w) \end{aligned}$$

Thus we get

$$\begin{aligned}
dv &= (-1)^{|v|+d+1} \iota_{H_v} w \\
&= (-1)^{|v|+d+1} \iota_{H_v} Dw \\
&= -(-1)^{|v|+d+1} (-1)^{(|v|+d-2)(|w|+d-2)} \{\!\!\{ w, v \}\!\!\}'_{\omega_\eta} \{\!\!\{ w, v \}\!\!\}''_{\omega_\eta} \\
&= -(-1)^{|v|+d+1} (-1)^{|v|+d} \{w, v\}_{\omega_\eta} \\
&= \{w, v\}_{\omega_\eta}
\end{aligned}$$

(we refer to (10.1) and (10.4) for the sign in the third line). Since $\{w, -\}_{\omega_\eta}$ is a derivation in its second argument we finally obtain for $f \in T_l V_c$

$$df = \{w, f\}_{\omega_\eta}$$

We must prove $\{w, w\}_{\omega_\eta} = 0$. Since $d^2 = 0$ we obtain as in (10.6) that $\{\{w, w\}_{\omega_\eta}, v\}_{\omega_\eta} = 0$ for all $v \in V$. Thus we must prove for $u \in T_l V_c$

$$\forall v \in V : \{u, v\}_{\omega_\eta} = 0 \Rightarrow u = 0$$

This is a linear statement so we may assume k is algebraically closed. It is also easy to see that it is invariant under Morita equivalence so we may pass to the quiver case. Then the statement follows immediately from a similar expression as (10.9).

Finally it remains to show that w contains only cubic terms and higher if $d_1 = 0$. This follows immediately from the explicit formula (11.10). \square

12. EXACT CALABI-YAU ALGEBRAS AND CYCLIC A_∞ -ALGEBRAS

In this section we assume that k has characteristic zero. Let A be a finite dimensional l - A_∞ -algebra. A A_∞ -cyclic structure of degree d on A is a symmetric bilinear form

$$(-, -) : A \times A \rightarrow \Sigma^d A$$

of degree d such that

$$(12.1) \quad (m_n(a_1, \dots, a_n), a_{n+1}) = (-1)^n (-1)^{|a_1|(|a_2|+\dots+|a_{n+1}|)} (m_n(a_2, \dots, a_{n+1}), a_1)$$

The following result can be used as an alternative approach to Lemma 11.3.1 which is part of the proof of Theorem 10.2.2 (see e.g. [23] for the relation).

Theorem 12.1. *Assume that k has characteristic zero. Let $A \in \text{PCAlg}(l)$ be homologically smooth and assume that the grading on A is concentrated in degrees ≤ 0 .*

Then the following statements are equivalent

- (1) $A^!$ has a finite dimensional minimal A_∞ -model (as augmented l - A_∞ -algebra, see A.3) with a cyclic A_∞ -structure of degree d .
- (2) A is exact d -Calabi-Yau.

We use the following technical lemma.

Lemma 12.2. *Let $W = l \oplus W_c \oplus lh$ be a finite dimensional minimal augmented l - A_∞ -algebra with h being l -central. Define $(-, -)$ as the composition*

$$(-, -) : W \times W \xrightarrow{m_2} W \xrightarrow{\text{projection}} lh \cong l \xrightarrow{\text{Tr}} k$$

Then $(-, -)$ defines a cyclic A_∞ structure if and only if

- (1) $(-, -)$ is a non-degenerate symmetric form.
- (2) The m_n for $n \geq 3$ have their image in W_c .

Proof. We first prove the \Rightarrow -direction. The non-degeneracy of $(-, -)$ and the fact that it is symmetric is by definition. Changing h by a non-zero scalar we may assume $(h, 1) = (1, h) = 1$.

The cyclic condition (12.1) for $n \geq 3$ gives for $u \in l$

$$(m_n(a_1, \dots, a_n), u) = \pm(m_n(u, a_1, \dots, a_{n-1}), a_n) = 0$$

Thus $m_n(a_1, \dots, a_n)$ must indeed be contained in W_c .

Now we prove the \Leftarrow -direction. We must prove (12.1) which simplifies to

$$(m_n(a_1, \dots, a_n), a_{n+1}) = (-1)^n (-1)^{|a_1|(2-n)} (a_1, m_n(a_2, \dots, a_{n+1}))$$

We write out the A_∞ -axiom for the m 's (see e.g. [18]), retaining only the terms which have a non-zero projection on lh . This yields

$$(-1)^n m_2(m_n(a_1, \dots, a_n), a_{n+1}) - (-1)^{|a_1|(2-n)} m_2(a_1, m_n(a_2, \dots, a_{n+1})) + \dots = 0$$

Taking the projection on lh gives what we want. \square

Proof of Theorem 12.1. We first prove (2) \Rightarrow (1). Thanks to Theorem 11.2.1 we know that A is weakly equivalent to $(T_l V, d)$ where V is as follows.

- (1) $d_1 : V \rightarrow V$ is zero.
- (2) $V = V_c \oplus lz$ with z an l -central element of degree $-d + 1$ and V_c finite dimensional.
- (3) V_c is concentrated in degrees $[-d + 2, 0]$.
- (4) $dz = \sigma' \eta \sigma''$ with $\eta \in (V_c \otimes_l V_c)_l$ being a non-degenerate and anti-symmetric element under β (cfr. §5.2).

It follows from Proposition A.5.4 that $W = l \oplus \Sigma^{-1} \mathbb{D}V = l \oplus W_c \oplus lh$ is isomorphic to $A^!$ as A_∞ -algebra where $h = s^{-1} z^*$ and $W_c = \Sigma^{-1} \mathbb{D}V_c$.

We define a symmetric l -bilinear form $(W \otimes_l W)_l \rightarrow k$ as follows: $(1, h) = (h, 1) = 1$, $(-, -)$ restricted to $W_c \times W_c$ is given by contraction with η (in the sense of (5.3)). All other values are zero.

By definition the $(m_n)_n$, restricted to $\bar{W} = W_c \oplus lh$ are dual to the components $(d_n)_n$ of the differential on $T_l V$. So we deduce from (4) that the m_n for $n \geq 3$ have their image in W_c . Furthermore the composition

$$W_c \times W_c \xrightarrow{m_2} W \xrightarrow{\text{projection}} lh \cong l \xrightarrow{\text{Tr}} k$$

is the bilinear form $(-, -)$. It now suffices to apply Proposition 12.2.

Now we prove the implication (1) \Rightarrow (2). It is an almost exact inversion of the above arguments. Let W be the augmented cyclic minimal model for $A^!$ and let $(-, -)$ be the associated symmetric non-degenerate l -bilinear form $(W \otimes_l W)_l \rightarrow k$ of degree d . We may write $\bar{W} = W_c \oplus lh$ where lh is dual to the augmentation. Thus h is l -central and we may assume $(h, 1) = (1, h) = 1$.

We deduce that $(-, -)$ restricts to a symmetric non-degenerate bilinear form on W_c . All other evaluations of $(-, -)$ on $W = l \oplus W_c \oplus lh$ are zero. For m_2 we find

$$(m_2(a, b), 1) = (m_2(1, a), b) = (a, b)$$

Thus the composition

$$W \times W \xrightarrow{m_2} W \xrightarrow{\text{projection}} lh \cong l \xrightarrow{\text{Tr}} k$$

coincides with $(-, -)$. From Proposition 12.2 we deduce that m_n has its image in W_c for $n \geq 3$.

Put $V = \Sigma \mathbb{D} \bar{W}$, $V_c = \Sigma \mathbb{D} W_c$. By Proposition A.5.4 A is weakly equivalent to $(T_l V, d)$. The symmetric bilinear form $(-, -)$ restricted to W_c must be given by contraction (in the sense of (5.4)) with some anti-symmetric element $\eta \in (V_c \otimes_l V_c)_l$ of degree $d - 2$.

Let $z = s^{-1}(h^*) \in V$. We must compute dz . In other words we must compute $d_n z$ which is the composition

$$(12.2) \quad lz \hookrightarrow V \hookrightarrow TV \xrightarrow{d} TV \xrightarrow{\text{projection}} V^{\otimes n}$$

Dually we must compute

$$(12.3) \quad W^{\otimes n} \hookrightarrow BW \xrightarrow{d} BW \xrightarrow{\text{projection}} lh$$

We have established that the image of (12.3) is zero when $n \geq 3$. For $n = 2$ is the bilinear form $(-, -)$ which is contraction with η (on \bar{W}). Dualizing this back to (12.2) we see that $d_n z = 0$ for $n \geq 3$ and $d_2 z = \sigma' \eta \sigma''$, \square

APPENDIX A. THE BAR COBAR FORMALISM

A.1. Weak equivalences. We survey the bar cobar formalism for subsequent dualization to the pseudo-compact case. We use [16, 19, 24, 26, 27] as modern references. We use some notations that were already introduced in §6.

If $C \in \text{Cog}(l)$, $A \in \text{Alg}(l)$ then

$$\text{Hom}_{l^e}(\bar{C}, \bar{A})$$

is a DG-vector space and the convolution product $*$ makes it into a DG-algebra. A *twisting cochain* is an element $\tau \in \text{Hom}_{l^e}(\bar{C}, \bar{A})_1$ satisfying the Maurer-Cartan equation

$$d\tau + \tau * \tau = 0$$

Let $\text{Tw}(C, A)$ denote the set of twisting cochains in $\text{Hom}_{l^e}(\bar{C}, \bar{A})$. It is easy to show that $\text{Tw}(-, A)$ is representable when restricted to complete augmented l -DG coalgebras. The representing object is called the *bar construction* on A and is denoted by BA . Likewise $\text{Tw}(C, -)$ is representable. The representing object is called the *cobar construction* on C and is denoted by ΩC . Thus we obtain natural isomorphisms

$$(A.1) \quad \text{Alg}(\Omega C, A) \cong \text{Tw}(C, A) \cong \text{Cog}(C, BA)$$

(the right one if C is cocomplete).

A weak equivalence between objects in $\text{Alg}(l)$ is defined to be a quasi-isomorphism. This naive definition does not work for coalgebras. A morphism $p : C \rightarrow C'$ in $\text{Cog}(l)$ is said to be a weak equivalence if $\Omega p : \Omega C \rightarrow \Omega C'$ is a quasi-isomorphism. This leads to the following result

Theorem A.1.1. [24, Thm 1.3.12]. *The functors (Ω, B) preserve weak equivalences and furthermore they define inverse equivalences between the categories $\text{Alg}(l)$ and $\text{Cog}(l)$, localized at weak equivalences.*

In particular we the counit/unit maps for (A.1)

$$(A.2) \quad \Omega BA \rightarrow A$$

$$(A.3) \quad C \rightarrow B\Omega C$$

are weak equivalences.

These weak equivalences are part of a model structure on $\text{Cogc}(l)$ which we will not fully specify. Let us mention however that every object is cofibrant and the fibrant objects are the l -DG-coalgebras which are cofree when forgetting the differential [24, §1.3].

A weak equivalence between augmented l -DG-coalgebras is a quasi-isomorphism but not necessarily the other way round (see [24, §1.3.5] for a counter example). This can be repaired in the following typical case.

Proposition A.1.2. [24, Prop. 1.3.5.1] *Assume the gradings on $C, C' \in \text{Cogc}(l)$ are concentrated in degrees ≥ 0 . Then a weak equivalence between C and C' is the same as a quasi-isomorphism.*

For completeness we recall the standard constructions of BA and ΩC . If V is a graded l -bimodule then the tensor algebra $T_l V = \bigoplus_{n \geq 0} V^{\otimes_l n}$ becomes in a natural way an augmented graded l -coalgebra if we put $\bar{T}_l V = \bigoplus_{n > 0} V^{\otimes_l n}$ and define the coproduct on $T_l V$ as

$$\Delta(v_1 | \cdots | v_n) = \sum_{i=0, \dots, n} (v_1 | \cdots | v_i) \otimes (v_{i+1} | \cdots | v_n)$$

where as customary $(v_1 | \cdots | v_n)$ denotes $v_1 \otimes \cdots \otimes v_n$ considered as an element of $V^{\otimes_l n} \subset T_l V$ and $() = 1$.

If A is an augmented l -DG-algebra then $BA = T_l(\Sigma \bar{A})$ with the codifferential d on $T_l(\Sigma \bar{A})$ being defined via its Taylor coefficients $d_n : (\Sigma \bar{A})^{\otimes n} \hookrightarrow T(\Sigma \bar{A}) \xrightarrow{d} T(\Sigma \bar{A}) \xrightarrow{\text{projection}} \Sigma A$

$$\begin{aligned} d_1(sa) &= -sda \\ d_2(sa|sb) &= (-1)^{|a|} s(ab) \\ d_n &= 0 \quad n \geq 3 \end{aligned} \tag{A.4}$$

for $a, b \in A$.

If C is a DG- l -coalgebra then $\Omega C = T_l(\Sigma^{-1} \bar{C})$ and the differential is given by

$$d(s^{-1}c) = -s^{-1}dc + (-1)^{|c(1)|} (s^{-1}c_{(1)} | s^{-1}c_{(2)}) \tag{A.5}$$

for $c \in C$.

A.2. Koszul duality. Let $A \in \text{Alg}(l)$. We recall the standard model structure on $\text{DGMod}(A)$.

- (1) The weak equivalences are the quasi-isomorphisms.
- (2) The fibrations are the surjective maps.
- (3) The cofibrations are the maps which have the left lifting property with respect to the acyclic fibrations.

It is possible to describe cofibrations more explicitly as retracts of standard cofibrations but we will not do it.

Now let $C \in \text{Cogc}(l)$. The following model structure is defined in [27, §8.2].

- (1) The weak equivalences are the morphisms with a coacyclic cone.
- (2) The fibrations are surjective morphisms with kernel which is injective when forgetting the differential.
- (3) The cofibrations are the injective morphisms.

An object is coacyclic if it is in the smallest subcategory of the homotopy category of C which contains total complexes of short exact sequences and is closed under arbitrary coproducts. This model structure looks different from the one defined in [24, §2.2.2]. However both model structures are Quillen equivalent to the one on $\mathrm{DGMod}(A)$ for $A = \Omega C$, defined above (see [24, Thm 2.2.2.2] and [27, §8.4]). So they have the same weak equivalences. Since they also have the same cofibrations they are the same.

We now discuss this Quillen equivalence. Let $M \in \mathrm{DGMod}(C^\circ)$ and $N \in \mathrm{DGMod}(A)$. Then $M \otimes_l N$ becomes a left DG-module over $\mathrm{Hom}_{l^e}(\bar{C}, \bar{A})$ if we let $\tau \in \mathrm{Hom}_{l^e}(\bar{C}, \bar{A})$ act by

$$\delta_\tau = (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \in \mathrm{End}(M \otimes_l N)$$

In particular if $\tau \in \mathrm{Tw}(C, A)$ then δ_τ satisfies the Maurer-Cartan equation in $\mathrm{End}(M \otimes_l N)$. We let $M \otimes_\tau N$ be equal to $M \otimes_l N$ but with δ_τ added to the differential.

There exists also an analogue of this construction in case $M \in \mathrm{DGMod}(A^\circ)$ and $N \in \mathrm{DGComod}(C)$. We leave the easy to guess formulas to the reader.

Here are some useful identities

$$\begin{aligned} (M \otimes_\tau A) \otimes_A N &= M \otimes_\tau N \\ M \square_C (C \otimes_\tau N) &= M \otimes_\tau N \end{aligned}$$

There is an analogue of the twisting construction for Hom . Let $M \in \mathrm{DGComod}(C)$ and $N \in \mathrm{DGMod}(A)$. Then $\mathrm{Hom}_l(M, N)$ becomes a left DG-module over $\mathrm{Hom}_{l^e}(\bar{C}, \bar{A})$ if we let $\tau \in \mathrm{Hom}_{l^e}(\bar{C}, \bar{A})$ act by

$$\delta_\tau(\phi) = \mu \circ (\tau \otimes \phi) \circ \Delta$$

If $\tau \in \mathrm{Tw}(C, A)$ then we let $\mathrm{Hom}_\tau(M, N)$ be equal to $\mathrm{Hom}_l(M, N)$ but with δ_τ added to the differential. Again this construction may also be performed with right (co)modules.

Now we have the following basic identities

$$\begin{aligned} \mathrm{Hom}_A(A \otimes_\tau M, N) &= \mathrm{Hom}_\tau(M, N) \\ \mathrm{Hom}_C(M, C \otimes_\tau N) &= \mathrm{Hom}_\tau(M, N) \end{aligned}$$

which yield a pair of adjoint functors [24, Theorem 2.2.2.2]

$$\begin{aligned} (A.6) \quad L : \mathrm{DGComod}(C) &\rightarrow \mathrm{DGMod}(A) : M \mapsto A \otimes_\tau M \\ R : \mathrm{DGMod}(A) &\rightarrow \mathrm{DGComod}(C) : N \mapsto C \otimes_\tau N \end{aligned}$$

Below we let τ_u be the twisting cochain $\bar{C} \rightarrow \overline{\Omega C}$ given by the obvious map. This is the universal twisting cochain corresponding to the identity map $\Omega C \rightarrow \Omega C$ in (A.1). In [27, §8.4] it is shown that in case $A = \Omega C$ and $\tau = \tau_u$ the adjoint pair (L, R) introduced above defines a Quillen equivalence. In particular a map $M \rightarrow N$ in $\mathrm{DGComod}(C)$ is a weak equivalence if and only if $\Omega C \otimes_\tau M \rightarrow \Omega C \otimes_\tau N$ is a quasi-isomorphism.

The following result is proved in a similar way as Proposition A.1.2.

Lemma A.2.1. *Assume that the grading on $C \in \mathrm{Cogc}(l)$ is concentrated in degrees ≥ 0 and $M, N \in \mathrm{DGComod}(C)$ are concentrated in degrees $\geq -n$ for certain n . Then a weak equivalence between M, N is the same as a quasi-isomorphism.*

A.3. A_∞ -algebras and minimal models. By definition a (non-unital) l - A_∞ -algebra is an l -bimodule A together with an l -coderivation d of degree one and square zero on the coalgebra $T_l(\Sigma A)$ compatible with the augmentation. By this we mean $d(1) = 0$, $\epsilon \circ d = 0$. We write $\tilde{B}A = (T_l(\Sigma A), d)$ and call $\tilde{B}A$ the bar construction of A . An A_∞ -morphism $A \rightarrow A'$ is a DG-coalgebra morphism $\tilde{B}A \rightarrow \tilde{B}A'$. We write $\text{Alg}_\infty^\bullet(l)$ for the category of l - A_∞ -algebras.

A coderivation on $T_l(\Sigma A)$ compatible with the augmentation is determined by “Taylor coefficients” ($n \geq 1$)

$$d_n : (\Sigma A)^{\otimes n} \hookrightarrow T_l(\Sigma A) \xrightarrow{d} T_l(\Sigma A) \xrightarrow{\text{projection}} \Sigma \bar{A}$$

which are of degree one. Introducing suitable signs the d_n may be transformed into maps

$$m_n : A^{\otimes n} \rightarrow A$$

of degree $2 - n$ (see e.g. [24, Lemme 1.2.2.1]). One has $m_1^2 = 0$, m_1 is a derivation for m_2 and m_2 is associative up to a homotopy given by m_3 . We view (A, m_1) as a complex and denote its homology by $H^*(A)$. In this way $(H^*(A), m_2)$ becomes a graded l -algebra (without unit).

Likewise an A_∞ -morphism $f : A \rightarrow A'$ is described by maps of degree $1 - n$

$$f_n : A^{\otimes n} \rightarrow A'$$

Here f_1 is a morphism of complexes $(A, m_1) \rightarrow (A', m'_1)$ which is compatible with the multiplications given by m_2 , m'_2 up to a homotopy given by f_2 . In particular $H^*(f_1)$ defines a morphism of graded l -algebras.

A morphism $f : A \rightarrow A'$ in $\text{Alg}_\infty^\bullet(l)$ is said to be a quasi-isomorphism (or weak equivalence) if $f_1 : (A, m_1) \rightarrow (A', m'_1)$ is a quasi-isomorphism.

The following is a basic result in the theory of A_∞ -algebras.

Proposition A.1. [24, Cor. 1.4.14] *Let $A \in \text{Alg}_\infty^\bullet(l)$ and let $(H^*(A), m_2)$ be its cohomology algebra. Then there exists a structure of an l - A_∞ -algebra on $H^*(A)$ of the form $(H^*(A), m_1 = 0, m_2, m_3, \dots)$ together with a morphism in $\text{Alg}_\infty^\bullet(l)$: $f : H^*(A) \rightarrow A$ which lifts the identity $H^*(A) \rightarrow H^*(A)$.*

An A_∞ -algebra with $m_1 = 0$ is said to be minimal. Following Kontsevich one calls the A_∞ -algebra $(H^*(A), m_1 = 0, m_2, m_3, \dots)$ a *minimal model* for A . It is unique up to non-unique isomorphism of l - A_∞ -algebras.

There is an obvious augmented version of the theory of A_∞ -algebras. An augmented l - A_∞ -algebra is an l - A_∞ -algebra A equipped with a decomposition of l -bimodules $A = l \oplus \bar{A}$ such that \bar{A} is a sub l - A_∞ -algebra of A and $1 \in l$ is a strict unit. I.e. $m_1(1) = 0$, $m_2(1, a) = a$, $m_2(a, 1) = a$ and $m_n(\dots, 1, \dots) = 0$ for $n \geq 3$. Note that the A_∞ -structure on A is completely determined by that of \bar{A} .

Likewise an morphism of augmented A_∞ -algebras $f : A \rightarrow A'$ is a morphism of l - A_∞ -algebras that restricts to a morphism of l - A_∞ -algebras $\bar{A} \rightarrow \bar{A}'$ such that $f_1(1) = 1$ and $f_n(\dots, 1, \dots) = 0$ for $n \geq 2$. Again f is completely determined by its restriction to \bar{A} . We denote the category of augmented l - A_∞ -algebras by $\text{Alg}_\infty(l)$.

For $A \in \text{Alg}_\infty(l)$ we put $BA = T_l(\Sigma \bar{A})$ and then the A_∞ -structure on \bar{A} defines a codifferential on BA compatible with the augmentation. Conversely augmented A_∞ -algebras may be defined in terms of codifferentials on $T_l(\Sigma \bar{A})$ which are compatible with the augmentation.

If A is an augmented l - A_∞ -algebra then there is a (natural) l - A_∞ -morphism $A \rightarrow \Omega BA$ to the DG-algebra ΩBA . This morphism is a quasi-isomorphism (see e.g. [24, Lemma 2.3.4.3]). The DG-algebra ΩBA is called the *DG-envelope* of A .

Lemma A.3.1. *If $A \rightarrow A'$ is an A_∞ -quasi-isomorphism then $BA \rightarrow BA'$ is a weak equivalence.*

Proof. We have to show that $\Omega BA \rightarrow \Omega BA'$ is a quasi-isomorphism. This follows from the fact that we have a commutative diagram

$$(A.7) \quad \begin{array}{ccc} A & \xrightarrow{\text{qi}} & A' \\ \text{qi} \downarrow & & \downarrow \text{qi} \\ \Omega BA & \longrightarrow & \Omega BA' \end{array}$$

□

Lemma A.3.2. *Assume that $C \in \text{Cogc}(l)$ is weakly equivalent to $(T_l V, d)$. Then there is an augmented l - A_∞ -quasi-isomorphism $l \oplus \Sigma^{-1}V \rightarrow \Omega C$.*

Proof. Note that giving the codifferential d on $T_l V$ is precisely the same as defining an augmented l - A_∞ -structure on $l + \Sigma^{-1}V$. As $(T_l V, d)$ is fibrant (see above) the weak equivalence $C \rightarrow T_l V$ is represented by an actual map of augmented l -DG-coalgebras. As $(T_l V, d) = B(l + \Sigma^{-1}V)$ we have the following quasi-isomorphisms

$$\Omega C \xrightarrow{DG} \Omega T_l V = \Omega B(l + \Sigma^{-1}V) \xleftarrow{A_\infty} l + \Sigma^{-1}V$$

The first map is in particular an augmented l - A_∞ -quasi-isomorphism so it can be inverted (e.g. [24, Cor. 1.3.1.3]). This yields what we want. □

A.4. The bar cobar formalism in the pseudo-compact case. In this paper we use the bar-cobar formalism in the context of pseudo-compact algebras and modules. To this end we simply dualize everything we have explained above, using \mathbb{D} . Let A, C be respectively objects in $\text{PCAlg}(l)$ and $\text{PCCog}(l)$. We put

$$(A.8) \quad \begin{aligned} BA &= \mathbb{D}\Omega\mathbb{D}A \\ \Omega C &= \mathbb{D}B\mathbb{D}C \end{aligned}$$

We may interpret these definitions more concretely. For $V \in \text{PCGr}(l)$ put

$$T_l V = \prod_{n \geq 0} V^{\otimes_l n}$$

One checks that $T_l V$ is naturally a graded augmented pseudo-compact l -algebra and coalgebra. Then $BA = T_l(\Sigma A)$, $\Omega C = T_l(\Sigma^{-1}C)$ with the differentials given by the formulas (A.4)(A.5).

We equip $\text{PCAlg}(l)$ with the dual model structure on $\text{Cogc}(l)$. In particular morphism $p : A \rightarrow A'$ in $\text{PCAlg}(l)$ is a weak equivalence if $Bp : BA \rightarrow BA'$ is a quasi-isomorphism. An object is cofibrant if it is of the form $(T_l V, d)$ with $V \in \text{PC}(l^e)$ and d compatible with the augmentation.

By similar dualizing we say that a weak equivalence between objects in $\text{PCCog}(C)$ is the same as a quasi-isomorphism.

We equip the categories $\text{PCDGComod}(C)$ and $\text{PCDGMod}(A)$ with the duals of the model structures on $\text{DGMod}(\mathbb{D}C^\circ)$ and $\text{DGComod}(\mathbb{D}A^\circ)$.

We dualize the functors L, R in the obvious way: $R = \mathbb{D}L\mathbb{D}$, $L = \mathbb{D}R\mathbb{D}$. They are given by the same formulas as (A.6) but now we use them with $C = BA$ and the universal (continuous) twisting cochain $\tau_u : \overline{BA} \rightarrow \bar{A}$.

A weak equivalence between objects in $\text{PCDGComod}(C)$ is the same as a quasi-isomorphism. On the other hand a morphism $M \rightarrow N$ in $\text{PCDGMod}(A)$ is a weak equivalence if and only if $BA \otimes_{\tau_u} M \rightarrow BA \otimes_{\tau_u} N$ is a quasi-isomorphism. The derived categories of A and C are obtained from $\text{PCDGMod}(A)$ and $\text{PCDGComod}(C)$ by inverting weak equivalences.

A.5. Minimal models for pseudo-compact algebras. If d is a differential on $T_l W$ with $W \in \text{PC}(l^e)$ then we will denote its components $W \rightarrow W^{\otimes n}$ by d_n .

We first note that since $\mathbb{D}T_l W \cong T_l(\mathbb{D}W)$, specifying a differential on $T_l W$ is exactly the same as specifying an augmented l - A_∞ -structure on $l + \Sigma^{-1}\mathbb{D}W$ (and this is an honest A_∞ -structure, not a pseudo-compact one).

For $A \in \text{PCAlgc}(l)$ we define the Koszul dual of A as (see also [19])

$$(A.9) \quad A^\dagger = \Omega \mathbb{D}A$$

Thus A^\dagger is an honest augmented l -DG-algebra (not a pseudo-compact DG-algebra).

Proposition A.5.1. (*Koszul duality, cfr [19]*) *There is an equivalence of triangulated categories*

$$D(A) \rightarrow D((A^\dagger)^\circ)^\circ$$

which sends $\Sigma^n l$ to $\Sigma^{-n} A^\dagger$.

Proof. We have

$$D(A) \cong D(BA) = D(\mathbb{D}\Omega \mathbb{D}A) \cong D((\Omega \mathbb{D}A)^\circ)^\circ$$

The functor realizing the indicated equivalence is given by

$$M \mapsto BA \otimes_{\tau_u} M \cong \mathbb{D}\Omega \mathbb{D}A \otimes_{\tau_u} M \cong \mathbb{D}(\mathbb{D}M \otimes_{\tau_u} A^\dagger) \mapsto \mathbb{D}M \otimes_{\tau_u} A^\dagger$$

We see that l is indeed sent to A^\dagger . □

Corollary A.5.2. *We have as algebras*

$$\text{Ext}_A^*(l, l) \cong H^*(A^\dagger)^\circ$$

Proof. We have

$$\begin{aligned} \text{Ext}_A^n(l, l) &= \text{Hom}_{D(A)}(l, \Sigma^n l) \\ &= \text{Hom}_{D(A^\dagger)^\circ}(A^\dagger, \Sigma^{-n} A^\dagger) \\ &= \text{Hom}_{D(A^\dagger)^\circ}(\Sigma^{-n} A^\dagger, A^\dagger) \\ &= A_n^\dagger \end{aligned}$$

One verifies that this identification inverts the order of the multiplication, whence the result. □

Remark A.5.3. One may show that A^\dagger actually computes $\text{RHom}_A(l, l)^\circ$.

Proposition A.5.4. *Let $A \in \text{PCAlgc}(l)$. Then there is a weak equivalence $\Omega \mathbb{D}A^\dagger \rightarrow A$. Furthermore the same holds with A^\dagger replaced by any augmented l - A_∞ -algebra quasi-isomorphic to it. Conversely if A is weakly equivalent to $(T_l W, d)$ then there is an A_∞ -quasi-isomorphism $l + \Sigma^{-1}\mathbb{D}W \cong A^\dagger$, where the A_∞ -algebra structure on $l + \Sigma^{-1}\mathbb{D}W$ is as introduced above.*

Proof. We have

$$\Omega \mathbb{D} A^! \cong \Omega \mathbb{D} \Omega \mathbb{D} A = \Omega B A$$

and $\Omega B A$ is weakly equivalent to A by applying \mathbb{D} to (A.3). This implies that A is weakly equivalent to $\Omega \mathbb{D} A^!$. The fact that may replace $A^!$ by any other algebra quasi-isomorphic to it follows from the fact that $\Omega \mathbb{D} A^! = \mathbb{D} B A^!$ combined with Lemma A.3.1.

Finally by applying \mathbb{D} to the conclusion of Lemma A.3.2 with $C = \mathbb{D} A$ and $V = \mathbb{D} W$ we obtain $A^! \cong l + \Sigma^{-1} \mathbb{D} W$. \square

Corollary A.5.5. *Let $A \in \text{PCAlg}(l)$. Then there exists a weak equivalence $(T_l W, d) \rightarrow A$ such that $d_1 = 0$.*

Proof. We let $l + \Sigma^{-1} W$ be a minimal augmented l - A_∞ -model for $A^!$. Then from Proposition A.5.4 obtain that A is weakly equivalent to $(T_l W, d)$ where $d_1 = 0$. \square

Following traditional terminology we call a weak equivalence as in Corollary A.5.5 a *minimal model* for A .

Corollary A.5.6. *Whenever we have a minimal model $T_l W \rightarrow A$ then $W \cong \Sigma^{-1}(\mathbb{D} \text{Ext}_A^*(l, l))_{\leq 0}$ and the m_2 multiplication on $l + \Sigma^{-1} \mathbb{D} W \cong \text{Ext}_A^*(l, l)$ for the induced A_∞ -structure corresponds to the opposite of the Yoneda multiplication on $\text{Ext}_A^*(l, l)$.*

Proof. By Proposition A.5.4 we have an A_∞ -quasi-isomorphism $l + \Sigma^{-1} \mathbb{D} W \cong A^!$ and hence an isomorphism as algebras

$$l + \Sigma^{-1} \mathbb{D} W \cong H^*(l + \Sigma^{-1} \mathbb{D} W) \cong H^*(A^!)$$

It now suffices to apply Corollary A.5.2. \square

APPENDIX B. HOCHSCHILD HOMOLOGY OF PSEUDO-COMPACT ALGEBRAS

Let $A \in \text{PCAlg}(l)$. It is easy to see that the tensor product $- \otimes_A -$ satisfies the hypotheses of [28, Prop. 4.1] in both arguments and hence it may be left derived in both arguments. It is also easy to see that deriving the first argument gives the same result as deriving the second argument. Therefore we make no distinction between the two and write the result as $- \overset{L}{\otimes}_A -$.

Now we work over A^e which is considered as an object in $\text{Mod}(l^e)$. Our aim is to show the following result

Proposition B.1. *If $A \in \text{PCAlg}(l)$ then $A \overset{L}{\otimes}_{A^e} A$ is computed by the standard Hochschild complex $(C(A), b) = ((A \otimes_l T_l(\Sigma A))_l, d_A + d_{\text{Hoch}})$ where d_{Hoch} is the usual Hochschild differential.*

Proof. In lemma B.2 below we show that $A \otimes_{\tau_u} B A \otimes_{\tau_u} A$ is a cofibrant replacement for A as in $\text{PCDGM}(A^e)$.

Thus we have the following formula

$$A \overset{L}{\otimes}_{A^e} A = A \otimes_{A^e} (A \otimes_{\tau_u} B A \otimes_{\tau_u} A)$$

We have

$$(B.1) \quad A \otimes_{A^e} (A \otimes_{\tau_u} B A \otimes_{\tau_u} A) \cong ((A \otimes_l T_l(\Sigma \bar{A}))^l, d_A + d_{\text{Hoch}})$$

where d_{Hoch} is the usual Hochschild differential.

The righthand side of (B.1) is the reduced Hochschild complex. It is quasi-isomorphic to the standard Hochschild complex which has the form

$$((A \otimes_l T_l(\Sigma A))^l, d_A + d_{\text{Hoch}})$$

(see [25, Prop. 1.6.5]). \square

Lemma B.2. $A \otimes_{\tau_u} BA \otimes_{\tau_u} A$ is a cofibrant replacement for A as in $\text{PCDGM}(\text{Mod}(A^e))$

Proof. We have a Quillen equivalence

$$(B.2) \quad \begin{aligned} L^e : \text{PCDGM}(\text{Comod}((BA)^e)) &\rightarrow \text{PCDGM}(\text{Mod}(A^e)) : N \mapsto A \otimes_{\tau_u} N \otimes_{\tau_u} A \\ R^e : \text{PCDGM}(\text{Mod}(A^e)) &\rightarrow \text{PCDGM}(\text{Comod}(BA^e)) : M \mapsto BA \otimes_{\tau_u} M \otimes_{\tau_u} BA \end{aligned}$$

As $A \otimes_{\tau_u} BA \otimes_{\tau_u} A$ is a projective bimodule when forgetting the differential it is cofibrant.

Hence we have to show that

$$\mu_{13} : A \otimes_{\tau_u} BA \otimes_{\tau_u} A \rightarrow A : a \otimes b \otimes c \mapsto a\epsilon(b)c$$

is a weak equivalence in $\text{PCDGM}(\text{Mod}(A^e))$. By the Quillen equivalence between $\text{PCDGM}(\text{Mod}(A^e))$ and $\text{PCDGM}(\text{Comod}(BA^e))$ we may as well show that the adjoint map

$$(B.3) \quad \Delta_{13} : BA \rightarrow BA \otimes_{\tau_u} A \otimes_{\tau_u} BA : c \mapsto c_{(1)} \otimes 1 \otimes c_{(2)}$$

is a weak equivalence, or equivalently a quasi-isomorphism of BA -bi-comodules. If we view (B.3) as a map of *left* comodules then it is precisely the unit map

$$BA \rightarrow RL(BA)$$

which is a weak equivalence (and hence quasi-isomorphism) since (L, R) forms a Quillen equivalence. \square

APPENDIX C. SYMMETRY FOR HOCHSCHILD HOMOLOGY

Assume that A is an l -algebra and let M be a finitely generated projective A -bimodule. Put $M^D = \text{Hom}_{A^e}(M, A \otimes A)$. Then an element $\xi \in M \otimes_{A^e} M$ defines a bimodule map

$$\xi^+ : M^D \rightarrow M : \phi \mapsto \phi(\xi')'' \xi' \phi(\xi')$$

and conversely using the identification

$$\text{Hom}_{A^e}(M^D, M) \cong M \otimes_{A^e} M$$

any bimodule morphism $M^D \rightarrow M$ is of the form ξ^+ for some $\xi \in M \otimes_{A^e} M$.

There is a $\mathbb{Z}/2\mathbb{Z} = \{1, \beta\}$ -action on $M \otimes_{A^e} M$ such that $\beta(a \otimes b) = b \otimes a$. One checks that

$$\beta(\xi)^+ = c \circ (\xi^+)^D$$

for the canonical isomorphism $c : M \mapsto M^{DD} : m \mapsto (\phi \mapsto \phi(m)'' \otimes \phi(m)')$

Hence if ξ is symmetric and we view c as an identification then $(\xi^+)^D = \xi^+$.

What we have just explained extends to the case where A is an l -DG-algebra and M is a perfect object in $D(A^e)$ (where we now use the derived version of $(-)^D$ as introduced in §8). We will apply it in the case $M = A$. We will prove the following result

Proposition C.1. $H_d(\beta)$ acts trivially on $\text{HH}_d(A) = H_d(A \overset{L}{\otimes}_{A^e} A)$. Hence if A is homologically smooth then any $\eta : A^D \rightarrow \Sigma^{-d}A$ is automatically self dual.

Proof. For our purpose we may and we will assume that A is cofibrant. We will use the complex $Y(A) = \Sigma\Omega_l^1 A \otimes \Sigma\Omega_l^1 A \oplus \Sigma(\Omega_l^1 A) \oplus \Sigma(\Omega_l^1 A)_l \oplus (A \otimes A)_l$ to compute $A \overset{L}{\otimes}_{A^e} A$ (see (11.5)).

Taking homology for rows and columns in $Y(A)$ we get two maps

$$l, r : Y(A) \rightarrow X(A)$$

where by a slight abuse of notation we have written $X(A)$ for $\text{cone}((\Omega_l^1 A)_{\natural} \xrightarrow{\partial_1} A_l)$ (see §7.2). Note that by Proposition 7.2.1 $X(A)$ computes the Hochschild homology of A .

We claim that l, r are homotopy equivalent. To prove this we will describe l and r explicitly:

$$\begin{aligned} l(s\omega_1 \otimes s\omega_2) &= 0 && (\text{on } \Sigma\Omega_l^1 A \otimes_{A^e} \Sigma\Omega_l^1 A) \\ l(s\omega_1) &= 0 && (\text{on the first copy of } (\Sigma\Omega_l^1 A)_l) \\ l(s\omega_2) &= s\omega_{2, \natural} && (\text{on the second copy of } \Sigma(\Omega_l^1 A)_l) \\ l(a \otimes b) &= \overline{ab} && (\text{on } (A \otimes_l A)_l) \end{aligned}$$

taking into account that in (11.4), $s\omega_1$ is represented by $s\omega_1 \otimes (1 \otimes 1)$, $s\omega_2$ is represented by $(1 \otimes 1) \otimes s\omega_2$, $a \otimes b$ is represented by $(a \otimes b) \otimes (1 \otimes 1)$ and taking homology for rows/columns corresponds to taking homology in the first/second factor. For the last line one needs to take into account the identification (11.6).

Likewise we have

$$\begin{aligned} r(s\omega_1 \otimes s\omega_2) &= 0 && (\text{on } \Sigma\Omega_l^1 A \otimes_{A^e} \Sigma\Omega_l^1 A) \\ r(s\omega_1) &= s\omega_{1, \natural} && (\text{on the first copy of } (\Sigma\Omega_l^1 A)_l) \\ r(s\omega_2) &= 0 && (\text{on the second copy of } \Sigma(\Omega_l^1 A)_l) \\ r(a \otimes b) &= (-1)^{|a||b|} \overline{ba} && (\text{on } (A \otimes_l A)_l) \end{aligned}$$

Thus for the difference $m = l - r$

$$\begin{aligned} m(s\omega_1 \otimes s\omega_2) &= 0 \\ m(s\omega_1) &= -s\omega_{1, \natural} \\ m(s\omega_2) &= s\omega_{2, \natural} \\ m(a \otimes b) &= \overline{[a, b]} \end{aligned}$$

Now we define a map of degree -1

$$h : Y(A) \rightarrow X(A)$$

by

$$\begin{aligned} h(s\omega_1 \otimes s\omega_2) &= 0 \\ h(s\omega_1) &= 0 \\ h(s\omega_2) &= 0 \\ h(a \otimes b) &= s(aDb)_{\natural} \end{aligned}$$

Now we compute $dh = [d, h] = \partial_1 \circ h + h \circ (\partial_1 \otimes 1 + 1 \otimes \partial_1)$. To this end we have to know ∂_1^{hor} and ∂_1^{ver} . We compute

$$\begin{aligned}\partial_1^{\text{hor}}(aDb) &= (\partial_1 \otimes 1)(aDb \otimes (1 \otimes 1)) \\ &= (ab \otimes 1 - a \otimes b) \otimes (1 \otimes 1) \\ &= ab \otimes 1 - a \otimes b\end{aligned}$$

$$\begin{aligned}\partial_1^{\text{ver}}(aDb) &= (1 \otimes \partial_1)((1 \otimes 1) \otimes aDb) \\ &= (1 \otimes 1) \otimes (ab \otimes 1 - a \otimes b) \\ &= 1 \otimes ab - (-1)^{|a||b|} b \otimes a\end{aligned}$$

taking into account the identification (11.6).

We now find

$$(dh)(s\omega_1 \otimes s\omega_2) = 0$$

$$\begin{aligned}(dh)(s(aDb)) &= h(ab \otimes 1 - a \otimes b) \\ &= -s(aDb)_{\mathfrak{h}} \quad (\text{on the first copy of } \Sigma\Omega_l^1 A)\end{aligned}$$

$$\begin{aligned}(dh)(s(aDb)) &= h((1 \otimes ab - (-1)^{|a||b|} b \otimes a) \\ &= s(Dab)_{\mathfrak{h}} - (-1)^{|a||b|} s(bDa)_{\mathfrak{h}} \\ &= s(aDb)_{\mathfrak{h}} \quad (\text{on the second copy of } \Sigma\Omega_l^1 A)\end{aligned}$$

$$\begin{aligned}(dh)(a \otimes b) &= \partial_1(s(aDb)_{\mathfrak{h}}) \\ &= \overline{[a, b]}\end{aligned}$$

Hence h is indeed a homotopy connecting l and r .

Now we have the following commutative diagram of complexes.

$$\begin{array}{ccc} Y(A) & \xrightarrow{\beta} & Y(A) \\ \downarrow l & & \downarrow r \\ X(A) & \xlongequal{\quad} & X(A) \end{array}$$

The top line comes from the fact that $Y(A)$ is obtained from tensoring the bimodule resolution of A with itself over A^e .

Taking homology we obtain

$$\begin{array}{ccc} H_d(Y(A)) & \xrightarrow{H_d(\beta)} & H_d(Y(A)) \\ \downarrow H_d(l) & & \downarrow H_d(r) \\ \text{HH}_d(A) & \xlongequal{\quad} & \text{HH}_d(A) \end{array}$$

Since r and l are homotopy we have $H_d(l) = H_d(r)$ and we are done. \square

APPENDIX D. KOSZUL DUALITY FOR HOCHSCHILD HOMOLOGY

The definition of the Hochschild mixed complex may be dualized to coalgebras. If C is a counital l -DG-coalgebra then the Hochschild mixed complex of C is $(C(C), b, B)$ where $(C(C), b)$ is the sum total complex of a double complex of the form

$$0 \rightarrow C^l \xrightarrow{\partial} (C \otimes_l C)^l \xrightarrow{\partial} (C \otimes_l C \otimes_l C)^l \rightarrow \dots$$

where l denotes the centralizer and ∂ is the dual of the Hochschild differential. B is the dual of the Connes differential. There exist a similar normalized mixed complex denoted by $(\bar{C}(C), b, B)$. The mixed complexes $\bar{C}(C)$ and $C(C)$ are quasi-isomorphic (see [25, Prop. 1.6.5]).

The following result is well known.

Proposition D.1. *Let $C \in \text{Cogc}(l)$. There is a quasi-isomorphism of mixed complexes.*

$$(D.1) \quad C(\Omega C) \rightarrow C(C)$$

Proof. Put $A = \Omega C$. Since C is cocomplete A is cofibrant. Hence we may apply Proposition 7.2.1 to obtain a quasi-isomorphism

$$C(A) \rightarrow MX(A)$$

By the dual version of [29, Theorem 4] we have an isomorphism of complexes

$$(\Omega_l^1 A)_{\natural} \cong \Sigma^{-1} C(\bar{C})$$

By definition $MX(A) = \text{cone}((\Omega_l^1 A)_{\natural} \xrightarrow{\partial_1} A_l)$. Since C is augmented we have $C = l \oplus \bar{C}$. We get isomorphisms as graded vector spaces

$$\begin{aligned} \text{cone}((\Omega_l^1 A)_{\natural} \xrightarrow{\partial_1} A_l) &= C(\bar{C}) \oplus A_l \\ &= C(\bar{C}) \oplus T_l(\Sigma^{-1} \bar{C}) \\ &= \bar{C}(C) \end{aligned}$$

One checks that this isomorphism is compatible with (b, B) and hence yields an isomorphism of mixed complexes

$$MX(A) \cong (\bar{C}(C), b, B)$$

Combining this with the standard quasi-isomorphism of mixed complexes (see [25, Prop. 1.6.5])

$$(\bar{C}(C), b, B) \rightarrow (C(C), b, B)$$

yields indeed a quasi-isomorphism as in (D.1). \square

Corollary D.2. *Let $A \in \text{PCAlgc}(l)$. Then we have a quasi-isomorphism of mixed complexes*

$$(D.2) \quad C(A^!) \rightarrow \mathbb{D}C(A)$$

This quasi-isomorphism is natural in A (taking into account that $A \mapsto A^!$ is a contravariant functor).

Proof. This follows the fact that $\mathbb{D}C(A) = C(\mathbb{D}A)$ together with Proposition D.1. The naturality of (D.2) follows from the naturality of (D.1). \square

Corollary D.3. *A weak equivalence $A \rightarrow A'$ in $\text{PCAlgc}(l)$ induces a quasi-isomorphism $C(A) \rightarrow C(A')$ of mixed complexes.*

Proof. By definition of $(-)^!$ we obtain that $A'^! \rightarrow A^!$ is a quasi-isomorphism in $\text{Alg}(l)$. Hence this induces a quasi-isomorphism $C(A'^!) \rightarrow C(A^!)$. By Corollary D.2 we get a commutative diagram of mixed complexes

$$\begin{array}{ccc} C(A'^!) & \xrightarrow{\cong} & \mathbb{D}C(A') \\ \downarrow \cong & & \downarrow \\ C(A^!) & \xrightarrow{\cong} & \mathbb{D}C(A) \end{array}$$

So the rightmost map is indeed a quasi-isomorphism. \square

Corollary D.4. *Assume that $A \rightarrow A'$ is a weak equivalence in $\text{PCAlg}(l)$ between homologically smooth algebras. Then A is exact d Calabi-Yau if and only if this is the case for A' .*

Proof. Taking into account Corollary D.3 we have to prove that $\eta \in \text{HH}_d(A)$ is non-degenerate if and only if its image in $\text{HH}_d(A')$ is non-degenerate. This is a formal verification which we leave to the reader. \square

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